Polar decomposition in metric spaces

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Contents

Definitions

Definitions

Polar coordinates

- For $d\mu(x) = \dot{\mu}(x)dx$ with $\dot{\mu} \in C^\infty(\mathbb{R}^n, \mathbb{R}_+)$,

$$\int_{\mathbb{R}^n} f(x)d\mu(x) = \int_0^\infty \int_{\mathcal{S}_r} f(r,\varphi)\omega(r,\varphi)d\varphi dr, \quad \forall f \in L^1_{\mathrm{loc}}(\mathbb{R}^n),$$

$$\mathcal{S}_r \doteq \{x \in \mathbb{R}^n | \quad |x| = r\}, \quad \omega(r,\varphi) = \frac{d\mu(x(r,\varphi))}{d\varphi dr}.$$

Definitions

An application

In

- M. Ruzhansky, D. Verma. “Hardy inequalities on metric measure spaces”. Proc. R. Soc. A 475:20180310, 2019
- M. Ruzhansky, D. Verma. “Hardy inequalities on metric measure spaces, II: the case $p > q$”. Proc. R. Soc. A 477:2021.0136, 2021

the authors assume

$$\int_{\mathcal{M}} f(x) d\mu(x) = \int_0^R \int_{\mathcal{S}_r(x_0)} f(x) d\omega_r(x) dr.$$

where $\mathcal{M}$ - metric space, μ-Borel measure.

Definitions

Definition

- $\mathcal{M}$ - metric space, μ - Borel measure on $\mathcal{M}$, $x_0 \in \mathcal{M}$
- $\rho_{x_0} \in C(\mathcal{M}, [0, R))$ given by

$$\rho_{x_0}(x) = d(x, x_0), \quad \forall x \in \mathcal{M} \tag{1}$$

- Spheres and balls of radius $r \in [0, R)$,

$$\mathcal{S}_r(x_0) \doteq \rho_{x_0}^{-1}(\{r\}), \quad \mathcal{B}_r(x_0) \doteq \rho_{x_0}^{-1}([0, r)) = \bigcup_{0 \leq s < r} \mathcal{S}_s(x_0)$$

- Borel measure $\nu_{x_0} \doteq \mu \circ \rho_{x_0}^{-1}$ on $[0, R)$,

$$\nu_{x_0}([0, r)) = \mu\left(\mathcal{B}_r(x_0)\right)$$

Definitions

Definition (cont.)

Definition

The metric measure space $(\mathcal{M}, d, \mu)$ admits a polar decomposition at $x_0 \in \mathcal{M}$ if $\exists r \mapsto \omega_r$ of bounded Borel measures ω_r on $\mathcal{M}$ for ν_{x_0}-a.e. $r \in [0, R)$, such that for every measurable $f : \mathcal{M} \to \mathbb{C}$, for which

$$\exists \int_{\mathcal{M}} f(x) d\mu(x),$$

the following conditions hold:

1. The map
$$[0, R) \ni r \mapsto \int_{\mathcal{M}} f(x) d\omega_r(x)$$
is Borel measurable.

Definitions

Definition (cont.)

Definition

2. The measure ω_r is concentrated on $\mathcal{S}_r(x_0)$ for ν_{x_0}-a.e. $r \in [0, R)$, i.e.,
$$\int_{\mathcal{M}} f(x) d\omega_r(x) = \int_{\mathcal{S}_r(x_0)} f(x) d\omega_r(x).$$
3. We have
$$\int_{\mathcal{M}} f(x) d\mu(x) = \int_{[0,R)} \int_{\mathcal{S}_r(x_0)} f(x) d\omega_r(x) d\nu_{x_0}(r)$$

Definitions

Assumptions

- Finite measure of balls, $\mu\left(\mathcal{B}_r(x_0)\right) < \infty$ for $r \in [0, R)$.
- Separability of $(\mathcal{M}, d)$ - not crucial.
- Completeness of $(\mathcal{M}, d)$ - not crucial.
- σ-compactness of $(\mathcal{M}, d)$ - sufficient and not far from necessary.

Proposition

Every locally finite Borel measure on a Polish space is concentrated on a σ-compact subspace, which is the union of a countable locally finite family of countable disjoint unions of compact sets.

General results

General results

The existence of a polar decomposition

Theorem

Let $(\mathcal{M}, d)$ be a separable metric space and $x_0 \in \mathcal{M}$. Let μ be a Borel measure on $(\mathcal{M}, \Sigma)$ such that:

1. *μ is inner regular (i.e., tight), that is,*
$$(\forall A \in \Sigma) \quad \mu(A) = \sup \{\mu(K) | \quad K \in \Sigma, \quad K \subset A, \quad K \text{ compact}\} .$$
2. *Balls $\mathcal{B}_r(x_0) \subset \mathcal{M}$ of any radius $r \in [0, R)$ have finite measure $\mu(\mathcal{B}_r(x_0)) < \infty$.*

Then $(\mathcal{M}, d, \mu)$ admits a polar decomposition at x_0.

General results

Absolute continuity of the measure ν_{x_0}

$$f_{x_0}(r) = \nu_{x_0}([0, r)) = \mu\left(\mathcal{B}_r(x_0)\right)$$

- ν_{x_0} a.c. w.r.t. the Lebesgue measure iff f_{x_0} a.c. on $[a, b] \subset [0, R)$
- ν_{x_0} a.c. implies

$$\int\limits_{[0,R)} \int\limits_{\mathcal{S}_r(x_0)} f(x) d\omega_r(x) d\nu_{x_0}(r) = \int\limits_0^R \int\limits_{\mathcal{S}_r(x_0)} f(x) \left[\frac{d\nu_{x_0}(r)}{dr} d\omega_r(x)\right] dr$$

Particular classes

Particular classes

Riemannian manifolds

Remark

Let $(\mathcal{M}, g)$ be a smooth connected Riemannian manifold with its geodesic distance d_g and Riemannian volume υ_g. For every ball $\mathcal{B}_r(x_0)$, if the Ricci curvature is bounded from below on $\mathcal{B}_r(x_0)$ then its Riemannian volume is finite, $\upsilon_g(\mathcal{B}_r(x_0)) < \infty$.

Proposition

Let $(\mathcal{M}, g)$ be a smooth connected Riemannian manifold with its geodesic distance d_g and Riemannian volume υ_g. Suppose that the Ricci curvature is bounded from below on every ball $\mathcal{B}_r(x_0)$. For every Borel measure μ on $\mathcal{M}$ that is absolutely continuous with respect to υ_g, and for every point x_0, the measure ν_{x_0} is absolutely continuous.

Particular classes

Riemannian manifolds (cont.)

Remark

The Ricci curvature is bounded globally on Riemannian manifolds with bounded geometry, including compact manifolds and homogeneous spaces. More generally, by Hopf-Rinow theorem, in a complete Riemannian manifold all balls are relatively compact, and thus the Ricci curvature is automatically bounded on each $\mathcal{B}_r(x_0)$*.*

Thus, on complete Riemannian manifolds all expectations are met. This is, however, not new.

Particular classes

Invariant sub-Riemannian structures on Lie groups

- G - connected real Lie group, μ - Radon measure
- $(G, \mathcal{D}, \langle , \rangle)$ - left-invariant sub-Riemannian structure
- d - the Carnot-Carathéodory metric

Then we have:

- (G, d) is complete as a metric space
- $\mathcal{B}_r(x_0)$ are relatively compact, and hence $\mu(\mathcal{B}_r(x_0)) < \infty$
- G is σ-compact, thus polar decomposition exists at $\forall x_0 \in G$

Question: Is ν_{x_0} absolutely continuous?

Thank you.

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