Elements (functions) that are universal with respect to a minimal system

Zhirayr Avetisyan

Ghent Analysis and PDE Center at University of Ghent, Belgium

Joint work with M. Grigoryan (Yerevan) and M. Ruzhansky (Ghent)

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Contents

Definitions

Definitions

Universal sequence

- Ω a Hausdorff sequential convergence space
- $\mathcal{X}$ a Banach space, with a continuous and dense embedding in $\mathcal{X} \hookrightarrow \Omega$
- $\{\varphi_n\}_{n=1}^{\infty}$ a minimal system in $\mathcal{X}$

Definition

A sequence $\{\xi_n\}_{n=1}^{\infty} \subset \mathcal{X}$ is called **universal** in Ω if subsequences of $\{\xi_n\}_{n=1}^{\infty}$ converge to every point of Ω,

$$(\forall f \in \Omega)\,(\exists \{N_k\}_{k=1}^{\infty} \subset \mathbb{N}) \quad \xi_{N_k} \xrightarrow[k\to\infty]{\Omega} f.$$

Definitions

Reductions

- $\mathcal{X} \hookrightarrow \Omega \hookrightarrow \Theta$ continuous dense embeddings
- $\mathcal{X}$ a Banach space
- Ω, Θ Hausdorff first-countable topological spaces

Lemma

Suppose that the sequence $\{\xi_n\}_{n=1}^{\infty} \subset \mathcal{X}$ is universal in Ω. Then it is universal in Θ.

Definitions

Minimal systems and series

- $\mathcal{X}$ a Banach space, $\mathcal{X}^*$ its dual space
- $\{\varphi_n\}_{n=1}^\infty \subset \mathcal{X}$ a minimal system
- $\{c_n\}_{n=1}^\infty \subset \mathcal{X}^*$ the dual system, $c_n(\varphi_m) = \delta_{n,m}$

Polynomial or series

$$\sum_{n=1}^{N} \alpha_n \varphi_n, \quad \alpha_n \in \mathbb{C}, \quad N \in \mathbb{N} \cup \{\infty\}$$

Fourier series of $f \in \mathcal{X}$

$$\sum_{n=1}^{\infty} c_n(f)\varphi_n$$

Definitions

Definition

A series $\sum_{n=1}^{\infty} \alpha_n \varphi_n$ is said to be:

1. **universal** in Ω in the **usual** sense if partial sums universal in Ω,

$$(\forall f \in \Omega)\,(\exists \{N_k\}_{k=1}^{\infty} \subset \mathbb{N}) \quad \sum_{n=1}^{N_k} \alpha_n \varphi_n \xrightarrow[k\to\infty]{} f$$

2. **universal** in Ω in the sense of $\mathbb{G}$-**values** (**signs**) if

$$(\forall f \in \Omega)\,(\exists \{\epsilon_n\}_{n=1}^{\infty}) \quad (\forall n \in \mathbb{N})\ \epsilon_n \in \mathbb{G} \quad \wedge \quad \sum_{n=1}^{\infty} \epsilon_n \alpha_n \varphi_n = f$$

3. **universal** in Ω in the sense of **rearrangements** if

$$(\forall f \in \Omega)\,(\exists \sigma \in \mathrm{Aut}(\mathbb{N})) \quad \sum_{n=1}^{\infty} \alpha_{\sigma(n)} \varphi_{\sigma(n)} = f$$

Definitions

Definition

We will call an element $U \in \mathcal{X}$:

1. **universal** for Ω w.r.t. $\{\varphi_n\}_{n=1}^{\infty}$ in the **usual** sense if the Fourier series $\sum_{n=1}^{\infty} c_n(U)\varphi_n$ is universal in Ω in the usual sense

2. **universal** for Ω w.r.t. $\{\varphi_n\}_{n=1}^{\infty}$ in the sense of $\mathbb{G}$-**values** (**signs**) if the Fourier series $\sum_{n=1}^{\infty} c_n(U)\varphi_n$ is universal in Ω in the sense of $\mathbb{G}$-values (signs)

3. **universal** for Ω w.r.t. $\{\varphi_n\}_{n=1}^{\infty}$ in the sense of **rearrangements** if the Fourier series $\sum_{n=1}^{\infty} c_n(U)\varphi_n$ is universal in Ω in the sense of rearrangements

Definitions

Definition

4. **conditionally universal** for Ω w.r.t. $\{\varphi_n\}_{n=1}^{\infty}$ if

$$(\exists\{\delta_n\}_{n=1}^{\infty}) \quad (\forall n \in \mathbb{N})\, \delta_n \in \mathbb{G}$$

and the series $\sum_{n=1}^{\infty} \delta_n c_n(U)\varphi_n$ is universal in Ω in the usual sense

5. **almost universal** for Ω w.r.t. $\{\varphi_n\}_{n=1}^{\infty}$ if

$$(\exists\{\delta_n\}_{n=1}^{\infty}) \quad (\forall n \in \mathbb{N})\, \delta_n \in \mathbb{G} \quad \wedge$$

$$\limsup_{n\to\infty} \frac{\#\{m \in \mathbb{N} | \quad m \leq n \wedge \delta_m = 1\}}{n} = 1$$

and the series $\sum_{n=1}^{\infty} \delta_n c_n(U)\varphi_n$ is universal in Ω in the usual sense

Universality in metric spaces

Universality in metric spaces

The Universal Approximation Property

Definition

Let $\Omega = (\Omega, d)$ be a metric space. We will say that the minimal system $\{\varphi_n\}_{n=1}^{\infty} \subset \mathcal{X}$ possesses the **universal approximation property** in Ω if for a dense subset $\mathcal{D} \subset \mathcal{X}$ we have

$$(\forall f \in \mathcal{D})\,(\forall \epsilon, \delta > 0)\,(\forall n_0 \in \mathbb{N})$$

$$(\exists N \in \mathbb{N})\left(\exists \{\alpha_n\}_{n=n_0}^{N} \subset \mathbb{C}\right)\left(\exists \{\delta_n\}_{n=n_0}^{N} \subset \mathbb{G}\right)$$

such that:

1. $\|H\|_{\mathcal{X}} < \epsilon$, $H \doteq \sum_{n=n_0}^{N} \alpha_n \varphi_n$
2. $d(f, Q) < \delta$, $Q \doteq \sum_{n=n_0}^{N} \delta_n \alpha_n \varphi_n$

Universality in metric spaces

Abstract theorem

- (Ω, d) a separable Abelian metric group
- $(\mathcal{X}, \|\cdot\|)$ a Banach space
- $\mathcal{X} \hookrightarrow \Omega$ a continuous dense additive embedding
- $(\{\varphi_n\}_{n=1}^{\infty}, \{c_n\}_{n=1}^{\infty})$ a bi-orthonormal system in $(\mathcal{X}, \mathcal{X}^*)$

Theorem

If the system $\{\varphi_n\}_{n=1}^{\infty}$ possesses the universal approximation property in Ω then $\exists U \in \mathcal{X}$ such that U is almost universal for Ω w.r.t. $\{\varphi_n\}_{n=1}^{\infty}$.

Asymptotic universality in L^1

Asymptotic universality in L^1

Definition

Let $\{\varphi_n\}_{n=1}^{\infty} \subset L^1(\mathcal{M})$ be a minimal system. $U \in L^1(\mathcal{M})$ is:

1. **asymptotically universal** for $L^1(\mathcal{M})$ w.r.t. $\{\varphi_n\}_{n=1}^{\infty}$ in the **usual** sense if there exists a sequence of subsets $\{F_m\}_{m=1}^{\infty} \subset 2^{\mathcal{M}}$, with

$$F_1 \subset F_2 \subset \ldots \subset \mathcal{M}, \quad \lim_{m\to\infty} \left| F_m^{\complement} \right| = 0, \tag{1}$$

such that

$$\left(\forall f \in L^1(\mathcal{M})\right) \left(\exists \{N_q\}_{q=1}^{\infty} \subset \mathbb{N}\right) (\forall m \in \mathbb{N})$$

$$\lim_{q\to\infty} \int_{F_m} \left| \sum_{n=1}^{N_q} c_n(U)\varphi_n(x) - f(x) \right| dx = 0$$

Asymptotic universality in L^1

Definition

2. **asymptotically conditionally universal** for $L^1(\mathcal{M})$ w.r.t. $\{\varphi_n\}_{n=1}^{\infty}$ if there exist a sequence of $\mathbb{G}$-values (signs) $\{\delta_n\}_{n=1}^{\infty}$, $(\forall n \in \mathbb{N})\, \delta_n \in \mathbb{G}$, and a sequence of subsets $\{F_m\}_{m=1}^{\infty} \subset 2^{\mathcal{M}}$, with

$$F_1 \subset F_2 \subset \ldots \subset \mathcal{M}, \quad \lim_{m\to\infty} \left| F_m^{\complement} \right| = 0, \tag{2}$$

such that

$$\left(\forall f \in L^1(\mathcal{M})\right) \left(\exists \{N_q\}_{q=1}^{\infty} \subset \mathbb{N}\right) (\forall m \in \mathbb{N})$$

$$\lim_{q\to\infty} \int_{F_m} \left| \sum_{n=1}^{N_q} \delta_n c_n(U) \varphi_n(x) - f(x) \right| dx = 0$$

Asymptotic universality in L^1

Definition

We will say that the minimal system $\{\varphi_n\}_{n=1}^{\infty} \subset L^1(\mathcal{M})$ possesses the **asymptotic approximation property** in $L^1(\mathcal{M})$ if for a dense subset $\mathcal{D} \subset L^1(\mathcal{M})$ and a positive number $C > 0$ we have

$$(\forall f \in \mathcal{D})\,(\forall \epsilon, \delta, \sigma > 0)\,(\forall n_0 \in \mathbb{N})$$
$$(\exists N \in \mathbb{N} + n_0)\left(\exists \{\alpha_n\}_{n=n_0}^{N} \subset \mathbb{C}\right)\left(\exists \{\delta_n\}_{n=n_0}^{N} \subset \mathbb{G}\right)(\exists E \subset \mathcal{M}) \qquad (3)$$

such that:

1. $\left|E^{\complement}\right| < \sigma$
2. $\|H\|_1 < \epsilon$, $H \doteq \sum_{n=n_0}^{N} \alpha_n \varphi_n$
3. $\int_E |f(x) - Q(x)|dx < \delta$, $Q \doteq \sum_{n=n_0}^{N} \delta_n \alpha_n \varphi_n$
4. $\int_{E^{\complement}} |Q(x)|dx \le C \cdot \|f\|_1$

Asymptotic universality in L^1

Existence of asymptotically conditionally universal functions

Theorem

If the minimal system $\{\varphi_n\}_{n=1}^{\infty}$ possesses the asymptotic approximation property in $L^1(\mathcal{M})$ then there exists an integrable function $U \in L^1(\mathcal{M})$ which is asymptotically conditionally universal for $L^1(\mathcal{M})$.

Asymptotic universality in L^1

Abundance of asymptotically conditionally universal functions

Theorem

If the minimal system $\{\varphi_n\}_{n=1}^{\infty}$ possesses the asymptotic approximation property in $L^1(\mathcal{M})$ then there exist an integrable function $U \in L^1(\mathcal{M})$, which is asymptotically conditionally universal for $L^1(\mathcal{M})$ w.r.t. $\{\varphi_n\}_{n=1}^{\infty}$, and a sequence of subsets $\{E_m\}_{m=1}^{\infty} \subset 2^{\mathcal{M}}$, with

$$E_1 \subset E_2 \subset \ldots \subset \mathcal{M}, \qquad \lim_{m\to\infty} \left| E_m^{\complement} \right| = 0,$$

such that

$$\left(\forall g \in L^1(\mathcal{M})\right) (\forall m \in \mathbb{N}) \left(\exists V_m \in L^1(\mathcal{M})\right)$$

$$(\forall x \in E_m) \quad V_m(x) = g(x)$$

$$\wedge \quad (\exists \{\varepsilon_n\}_{n=1}^{\infty} \subset \mathbb{G})(\forall n \in \mathbb{N}) \quad c_n(V_m) = \varepsilon_n c_n(U).$$

Examples

Examples

The trigonometric system in $L^p([-\pi,\pi])$, $p \in [0,1)$

Lemma (Galoyan, Grigoryan'21)

Let $\Omega = L^p([-\pi,\pi])$, $p \in (0,1)$, $\mathcal{X} = L^1([-\pi,\pi])$ and consider the trigonometric system

$$\varphi_n(x) = e^{\imath nx}, \quad \forall x \in [-\pi,\pi], \quad \forall n \in \mathbb{N}.$$

Then the system $\{\varphi_n\}_{n=1}^{\infty}$ possesses the universal approximation property in Ω.

Corollary

There exists a function $U \in L^1([-\pi,\pi])$ which is almost universal for $L^p([-\pi,\pi])$, $p \in [0,1)$, as well as $M([-\pi,\pi])$, w.r.t. the trigonometric system.

Examples

The trigonometric system in $L^1([-\pi,\pi])$

Lemma (Galoyan, Grigoryan'18)

The trigonometric system $\{\varphi_n\}_{n=1}^{\infty}$ possesses the asymptotic approximation property in $L^1([-\pi,\pi])$.

Corollary

There exists a function $U \in L^1([-\pi,\pi])$ which is asymptotically conditionally universal for $L^1([-\pi,\pi])$ w.r.t. the trigonometric system.

Corollary

For every $\epsilon > 0$ there exists a set E with $|E^{\complement}| < \epsilon$, such that for every function $f \in L^1([-\pi,\pi])$ one can find a function $U \in L^1([-\pi,\pi])$ with $U|_E = f|_E$, which is asymptotically conditionally universal for $L^1([-\pi,\pi])$ w.r.t. the trigonometric system.

Conclusion

Main results

- Almost universal elements in metric spaces
- Asymptotically conditionally universal functions in L^1

Conclusion

Open questions

- The universal approximation property of various systems
- The asymptotic approximation property of various systems

Thank you.

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