Approximations in L^1 with convergent Fourier series.

Zhirayr Avetisyan

joint work with Martin Grigoryan (Yerevan) and Michael Ruzhansky (London)

Department of Mathematics, UCSB

Applied/PDE Seminar, UCSB 2018

Contents

Measure space

Approximations

- Norm approximations: given f find g s.t.

$$\|f - g\| < \epsilon$$

- Spatial approximations: given f find g s.t.

$$(f - g)|_E = 0, \quad |E^{\complement}| < \epsilon$$

Theorem (Luzin'1912)

For every $f : [a, b] \to \mathbb{C}$ measurable, a.e. finite, and $\epsilon > 0$ there exist $g \in C[a, b]$ and $E_{f,\epsilon} \subset [a, b]$ s. t.

$$(f - g)|_{E_{f,\epsilon}} = 0, \quad |E_{f,\epsilon}| > b - a - \epsilon.$$

Fourier series in L^1

- $\mathcal{M} = (\mathcal{M}, \Sigma, \mu)$ finite separable measure space
- $\{\varphi_n\}_{n=1}^{\infty}$ ONB in $L^2(\mathcal{M})$, $\quad \varphi_n \in L^\infty(\mathcal{M})$ for all $n \in \mathbb{N}$
- For $f \in L^1(\mathcal{M})$,

$$c_n(f) = (f, \varphi_n)_2 = \int_{\mathcal{M}} f(x)\varphi_n^*(x)d\mu(x), \quad Y_n(x; f) = c_n(f)\varphi_n(x)$$

- **Problem:** For which $f \in L^1(\mathcal{M})$ does the Fourier series converge?

$$\sum_{n=1}^{\infty} Y_n(.; f) \to f \quad \text{in} \quad L^1(\mathcal{M})$$

Spatial approximations in L^1

Theorem (Grigoryan'91)

Let

$$\varphi_n(x) = e^{\imath nx} \in L^\infty[-\pi,\pi].$$

For $\forall\epsilon > 0$ there is a subset

$$E_\epsilon \subset [-\pi,\pi] \quad \text{with} \quad |E_\epsilon| > 2\pi - \epsilon$$

with the following property; for every $f \in L^1[-\pi,\pi]$ there is a $g \in L^1[-\pi,\pi]$ such that

$$(f-g)|_{E_\epsilon} = 0 \quad \text{and} \quad \sum_n Y_n(.,g) \to g \quad \text{in} \quad L^1[-\pi,\pi].$$

Cylindric structure

Obstacles against generalization to $\{\varphi_n\}_{n=1}^{\infty}$ and $\mathcal{M}$:

- Uses special properties of $e^{\imath nx}$
- Uses the structure of $[a, b] \subset \mathbb{R}$

Possibilities:

- Use measure theoretic isomorphism $\mathcal{M} \simeq [a, b]$
- Find natural cylindric structure $\mathcal{M} = [a, b] \times \mathcal{N}$

The main result

Theorem:

Let $\mathcal{M}$ be a separable finite diffuse measure space, and let $\{\varphi_n\}_{n=1}^{\infty}$ be an ONB in $L^2(\mathcal{M})$ with $\varphi_n \in L^\infty(\mathcal{M})$. For every $\epsilon, \delta > 0$ there is a subset $E_\epsilon \subset \mathcal{M}$ with $|E_\epsilon| > |M| - \delta$ and with the following property; for each function $f \in L^1(\mathcal{M})$ with $|f|_1 > 0$ there exists a $g \in L^1(\mathcal{M})$ s.t.

- $\|f - g\|_1 < \epsilon$,
- $(f - g)|_{E_\epsilon} = 0$,
- $$\sum_{n=1}^{\infty} Y_n(.; g) \to g \in L^1(\mathcal{M}),$$
- $$\sup_m \left\| \sum_{n=1}^{m} Y_n(.; g) \right\|_1 < 2 \min\{\|f\|_1, \|g\|_1\}.$$

Homogeneous Space

Example: sphere

- $\mathcal{M} = \mathbb{S}^2$, $x = (\theta, \varphi)$,

$$d\mu(x) = \frac{1}{4\pi^2} \sin\theta d\theta d\varphi.$$

- Label $n = (\rho, m)$, $\quad \rho \in \mathbb{N}_0, \quad m = -\rho, \ldots, \rho$,

$$\varphi_n(x) = Y_\rho^m(\theta, \varphi) = \sqrt{\frac{(2\rho+1)(\rho-m)!}{(\rho+m)!}} P_\rho^m(\cos\theta) e^{\imath m \varphi}.$$

- Cylindric structure $\mathbb{S}^2 \simeq [0, 1] \times [0, 1]$,

$$(\theta, \varphi) \mapsto (\frac{\varphi}{2\pi}, \frac{\theta}{\pi}).$$

Invariant measure on G/H

- G - infinite compact second countable Hausdorff group
- $H \subset G$ - closed subgroup, G/H - infinite homogeneous space
- $x \in G/H$ means $x = \mathrm{x}H$, where $\mathrm{x} \in G$
- $d\mathrm{x}$ - Haar measure on G, dh - Haar measure on H
- dx - unique normalized G-invariant Radon measure on G/H such that
$$\int_G f(\mathrm{x})d\mathrm{x} = \int_{G/H} \left(\int_H f(\mathrm{x}h)dh \right) dx, \quad \forall f \in C(G).$$

Fourier transform on G/H

- $\hat{G}$ - unitary dual of G
- $d_\rho^H = \text{mult}(\mathbf{1}, \rho|_H)$ for $\forall \rho \in \hat{G}$
- $$\widehat{G/H} = \left\{\rho \in \hat{G} \middle| \quad d_\rho^H > 0\right\}$$
- $$\mathbb{P}_H = \int_H \rho(h) dh, \quad \rho(x) = \rho(\mathrm{x})\mathbb{P}_H, \quad \forall \rho \in \widehat{G/H}, \quad \forall x = \mathrm{x}H \in G/H$$
- **Fourier** transform of $f \in L^1(G/H)$ is
$$\hat{f}(\rho) = \int_{G/H} f(x)\rho(x)^* d\mu(x)$$

Fourier series on G/H

- Block Fourier series of $f \in L^1(G/H)$,

$$\sum_{\rho \in \widehat{G/H}} d_\rho \operatorname{tr}\left[\hat{f}(\rho)\rho(x)\right], \quad d_\rho = \dim \mathcal{H}_0\rho \in \mathbb{N}$$

- Choose $\{e_i^\rho\}_{i=1}^{d_\rho}$ ONB in $\mathcal{H}_\rho$, and $\{e_\alpha^\rho\}_{\alpha=1}^{d_\rho^H}$ ONB in $\mathbb{P}_H\mathcal{H}_\rho$

- Label $n = (\rho, i, \alpha)$, $\quad \varphi_n(x) = \sqrt{d_\rho}\left(\rho(x)e_\alpha^\rho, e_i^\rho\right)$,

$$c_{\rho,i,\alpha}(f) = (f, \varphi_{\rho,i,\alpha})_2, \quad Y_{\rho,i,\alpha}(x;\rho) = c_{\rho,i,\alpha}(f)\varphi_{\rho,i,\alpha}(x)$$

- Then

$$\operatorname{tr}\left[\hat{f}(\rho)\rho(x)\right] = \sum_{i=1}^{d_\rho} \sum_{\alpha=1}^{d_\rho^H} Y_{\rho,i,\alpha}(x;\rho)$$

Main theorem for G/H

Theorem:

Let $\mathcal{M} = G/H$ be an infinite homogeneous space of a compact second countable Hausdorff group G with closed isotropy subgroup $H \subset G$. For every $\epsilon, \delta > 0$ there is a subset $E_\epsilon \subset G/H$ with $|E_\epsilon| > 1 - \delta$ and with the following property; for each function $f \in L^1(G/H)$ with $|f|_1 > 0$ there exists a $g \in L^1(G/H)$ s.t.

- $\|f - g\|_1 < \epsilon$,
- $(f - g)|_{E_\epsilon} = 0$,
- $$\sum_{\rho \in \widehat{G/H}} d_\rho \operatorname{tr}[\hat{g}(\rho)\rho(x)] \to g \in L^1(G/H),$$
- $$\sup_{\rho_0 \in \widehat{G/H}} \left\| \sum_{\rho \leq \rho_0} d_\rho \operatorname{tr}[\hat{g}(\rho)\rho(x)] \right\|_1 < 2 \min\{\|f\|_1, \|g\|_1\}.$$

Cylindric structure in G/H

- $K \subset G$ infinite closed subgroup,

$$G/H^{(K)} = \{x \in G/H| \quad \exists \mathbf{1} \neq k \in K \quad \text{s.t.} \quad kx = x\} \subset G/H$$

- $\mathrm{q} : G/H \to K\backslash G/H, \quad \mathrm{q}(x) = Kx$

- $\nu = \mu \circ \mathrm{q}^{-1}$ probability measure on $K\backslash G/H$

Proposition

If $|G/H^{(K)}| = 0$ then there exists a Borel almost isomorphism $\varphi : K \times K\backslash G/H \to G/H$ such that

$$\varphi(\tilde{k} \cdot k, Kx) = \tilde{k}\varphi(k, Kx), \quad \mathrm{q}(\varphi(k, Kx)) = Kx,$$

$$\forall k, \tilde{k} \in K, \quad \forall Kx \in K\backslash G/H.$$

Construction

The construction of E_ϵ and g

- Choose an arbitrary ordering $\{R_k\}_{k=1}^{\infty}$ of all Fourier polynomials with rational coefficients.

- For every $k \in \mathbb{N}$, choose a partition $\{\Delta_l(k)\}_{l=1}^{\nu_0(k)}$ of the cylindric measure space $\mathcal{M} = [0,1] \times \mathcal{N}$ of the form

$$\Delta_l(k) = [a_l(k), b_l(k)] \times \tilde{\Delta}_l(k),$$

such that the measures $\|\Delta_l(k)\|$ are small enough, as well as a subordinate real step function

$$\Lambda(k) = \sum_{l=1}^{\nu_0(k)} \gamma_l(k) \chi_{\Delta_l(k)},$$

such that $\|\Lambda(k) - R_k\|_1$ is sufficiently small.

The construction of E_ϵ and g (2)

- For every $k \in \mathbb{N}$, choose a number $\delta_*(k) \in \left(0, \frac{1}{2}\right)$ so that $\{\delta_*(k)\}_{k=1}^{\infty}$ decays sufficiently rapidly. Define the periodic step function

$$I(t) = 1 - \frac{1}{\delta_*(k)} \chi_{[0,\delta_*(k))}(t \mod 1),$$

and the measurable function $\hat{g}_l^k \in L^\infty(\mathcal{M})$ by

$$g_l^k(x) = \gamma_l(k) I\big(s_0(k)t\big) \chi_{\Delta_l(k)}(x), \quad \forall x = (t, y) \in \mathcal{M} = [0, 1] \times \mathcal{N},$$

where the positive number $s_0(k)$ is sufficiently large.

The construction of E_ϵ and g (3)

- Define the measurable subsets $\hat{E}_l(k) \subset \Delta_l(k)$ by

$$\hat{E}_l(k) = \left\{ x \in \Delta_l(k) \middle| \quad g_l^k(x) = \gamma_l(k) \right\}.$$

Define inductively the natural numbers $\hat{N}_l(k)$, $l = 0, \ldots, \nu_0(k)$ and Fourier polynomials $\hat{Q}_l^k$, $l = 1, \ldots, \nu_0(k)$, by setting $\hat{N}_0(1) = 1$, $\hat{N}_0(k) = \hat{N}_{\nu_0(k)}(k-1)$ for $k > 1$, and $\hat{Q}_l^k = \sum_{n=\hat{N}_{l-1}(k)}^{\hat{N}_l(k)-1} Y_n(\hat{g}_l^k)$, so that the quantities

$$\left\| \sum_{n=1}^{\hat{N}_l(k)-1} Y_n(\hat{g}_l^k) - \hat{g}_l^k \right\|_2$$

are sufficiently small.

The construction of E_ϵ and g (4)

- For every $k \in \mathbb{N}$, define the natural numbers $N_k = \hat{N}_{\nu_0(k)} - 1$, measurable subsets $\tilde{E}_k = \bigcup_{l=1}^{\nu_0(k)} \hat{E}_l(k)$, and measurable functions $\tilde{g}_k \in L^\infty(\mathcal{M})$ by

$$\tilde{g}_k = R_k - \Lambda(k) + \sum_{l=1}^{\nu_0(k)} \hat{g}_l^k,$$

  as well as Fourier polynomials

$$\tilde{Q}_k = \sum_{l=1}^{\nu_0(k)} \hat{Q}_l^k = \sum_{n=N_{k-1}}^{N_k - 1} \tilde{Y}_n.$$

- Set

$$E = \bigcap_{k=1}^{\infty} \tilde{E}_k.$$

The construction of E_ϵ and g (5)

- Choose a subsequence $\{R_{k_s}\}_{s=0}^{\infty}$ such that $\|R_{k_s}\|_1$ decay sufficiently rapidly, and $\sum_{s=0}^{\infty} R_{k_s} = f$ in $L^1(\mathcal{M})$.
- Define inductively the sequence of natural numbers $\{\nu_s\}_{s=1}^{\infty}$, $\nu_s > \nu_{s-1}$ for $s > 1$, and measurable functions $g_s \in L^\infty(\mathcal{M})$ by choosing ν_1 so that $N_{\nu_1 - 1} > \max \sigma(R_{k_0})$ and

$$\left\| R_{\nu_s} - R_{k_s} + \sum_{j=1}^{s-1} \left[\tilde{Q}_{\nu_j} - g_j \right] \right\|_1$$

  is sufficiently small, and setting $g_s = R_{ks} + \tilde{g}_{\nu_s} - R_{\nu_s}$.
- Finally, set

$$g = R_{k_0} + \sum_{s=1}^{\infty} g_s.$$

Thank you.