Explicit spectral asymptotics of a first order elliptic system on a manifold

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Weyl quantization on a manifold

- M - closed (i.e., compact without boundary) connected C^∞ manifold, $\dim M = n \geq 2$
- $\mathrm{Vol}^{1/2} M$ - half-density bundle over M
- $A \in \mathrm{L}^d(\mathrm{Vol}^{1/2} M)$ is a ΨDO of order $d \in \mathbb{R}$, i.e.,
- $A : C^\infty(\mathrm{Vol}^{1/2} M) \to C^\infty(\mathrm{Vol}^{1/2} M)$ s.t. $\forall$ chart $M \subset \Omega \simeq \Omega' \subset \mathbb{R}^n$,

$$A\Big|_{C_c^\infty(\mathrm{Vol}^{1/2}\Omega)} = \mathrm{Op}(a_\Omega) + K_\Omega, \quad K_\Omega(.,.) \in C^\infty(\mathrm{Vol}^{1/2} M \boxtimes \mathrm{Vol}^{1/2} M).$$

$$a_\Omega \in \mathrm{S}^d(\mathrm{T}^*\Omega).$$

The setting of the problem

Weyl quantization on a manifold (cont.)

- $M \supset \Omega_i \simeq \Omega_i' \subset \mathbb{R}^n$, $i = 1, 2$, two charts, $\varphi : \Omega_1 \cap \Omega_2 \to \Omega_1 \cap \Omega_2$ transition map,

$$a_{\Omega_1} = \varphi^* a_{\Omega_2} \mod \mathrm{S}^{d-2}.$$

- Isomorphism $\mathrm{S}^d/\mathrm{S}^{d-2} \ni a \mapsto \mathrm{Op}(a) \in \mathrm{L}^d/\mathrm{L}^{d-2}$
- a_d - principal symbol, a_{d-1} - subprincipal symbol,

$$a = a_d + a_{d-1} \mod \mathrm{S}^{d-2}.$$

The setting of the problem

First order system on a closed manifold

- $\mathbf{H} = \bigoplus_{1}^{m} L^2(\mathrm{Vol}^{1/2} M)$ - intrinsic L^2-space of m-columns
- $A : \bigoplus_{1}^{m} C^\infty(\mathrm{Vol}^{1/2} M) \to \bigoplus_{1}^{m} C^\infty(\mathrm{Vol}^{1/2} M)$ <u>symmetric</u> w. r. t. $\mathbf{H}$
- $A \in \mathrm{L}^1(\bigoplus_{1}^{m} \mathrm{Vol}^{1/2} M)$, a_1, a_0 Hermitian $m \times m$ matrices

$$A = \mathrm{Op}(a_1 + a_0) \mod \mathrm{L}^{-1}(\bigoplus_{1}^{m} \mathrm{Vol}^{1/2} M)$$

The setting of the problem

Elliptic and multiplicity-free

- A elliptic, i.e., $\det a_1(p) \neq 0, \quad \forall p \in \mathrm{T}^* M$
- a_1 has simple spectrum, $\sharp\sigma(a_1(p)) = m, \quad \forall p \in \mathrm{T}^* M$
- $M \supset \Omega \simeq \Omega' \subset \mathbb{R}^n$ local chart, $\mathrm{T}^*\Omega \ni p = (x, \xi) \in \Omega \times \mathbb{R}^n$

The setting of the problem

Spectrum of A

- A self-adjoint with domain $\mathbf{D} = \bigoplus_{1}^{m} H^1(\mathrm{Vol}^{1/2} M)$
- $\sigma(A)$ discrete
- $\sup \sigma(A) < +\infty \quad \text{iff} \quad a_1 \leq 0$
- $\inf \sigma(A) > -\infty \quad \text{iff} \quad a_1 \geq 0$

The setting of the problem

Eigenvalues and eigenfunctions

- Eigenvalues $\sigma(a_1(p)) = \{h^j(p)\}, \quad j = -m^-, \ldots, -1, 1, \ldots, m^+$
- $m^+ + m^- = m, \quad j \cdot h^j(p) > 0, \quad \forall j, \quad \forall p \in \mathrm{T}^*M$
- Eigenvectors

$$a_1(p)v^j(p) = h^j(p)v^j(p), \quad v^j \in C^\infty(\mathrm{T}^*M, \mathbb{C}^m)$$

- Eigenprojections

$$P^j(p) = v^j(p) \times [v^j(p)]^*, \quad P^j \in C^\infty(\mathrm{T}^*M, \mathrm{gl}(m))$$

The setting of the problem

Hamiltonian dynamics

- $h^j(p) = h^j(x, \xi)$ - Hamiltonian
- Trajectory $\quad \big(\, x^j(t; y^j, \eta^j)\,,\, \xi(t; y^j, \eta^j)\,\big)$

$$\frac{dx(t)}{dt} = \partial_\xi h^j(x, \xi), \qquad \frac{d\xi(t)}{dt} = -\partial_x h^j(x, \xi)$$

$$(\, x^j(0; y^j, \eta^j)\,,\, \xi(0; y^j, \eta^j)\,) = (y^j\,,\, \eta^j)$$

The setting of the problem

Local counting functions

- Spectrum $\sigma(A) = \{\lambda_k\}, \quad k \in -\mathbb{N} \cup \mathbb{N}$
- $|\lambda_k| \nearrow$ as $|k| \nearrow, \quad k \cdot \lambda_k \geq 0$
- Eigenfunctions $Av_k = \lambda_k v_k$, $v_k \in \mathbf{D}$
- Local counting functions $N_\pm \in C(\mathrm{Vol}M) \otimes \mathcal{D}(\mathbb{R})'$

$$N_\pm(x, \lambda) = \begin{cases} 0 & \text{if} \quad \lambda \leq 0 \\ \sum\limits_{0 < \pm\lambda_k < \lambda} \|v_k(x)\|^2 \end{cases}$$

- $N_\pm(x, \lambda) \nearrow$ as $\lambda \nearrow$ (step function)

The setting of the problem

Spectral aymptotics

- $y \in M$ *non-focal* if

$$\left|\left\{\eta \in \mathrm{T}^*_y M \,\middle|\, \|\eta\| = 1, \quad \exists T > 0 \quad \text{s. t.} \quad x^j(T; y, \eta) = y\right\}\right| = 0, \quad \forall j$$

Theorem (classical)

If $x \in M$ is non-focal then uniformly in $x \in M$

$$N_\pm(\lambda, x) = \lambda^n \frac{c^\pm_{n-1}(x)}{n} + \lambda^{n-1} \frac{c^\pm_{n-2}(x)}{n-1} + \mathrm{o}(\lambda^{n-1}).$$

$c^\pm_{n-1}, c^\pm_{n-2} \in C(\mathrm{Vol} M)$

The setting of the problem

Mollified spectral aymptotics

- $\rho \in \mathcal{S}(\mathbb{R})$ s. t. $\hat{\rho} \in C_c^\infty(-\mathbf{T}, \mathbf{T})$, $\quad \hat{\rho}(0) = 1$, $\quad \rho'(0) = 0$,

$$\mathbf{T} = \inf \left\{ T > 0 \middle| \quad \exists (y, \eta) \in \mathrm{T}^* M,\ \exists j \quad \text{s. t.} \quad x^j(T; y, \eta) = y \right\} > 0.$$

Theorem (classical)

Uniformly in $x \in M$

$$[N_\pm * \rho](x, \lambda) = \lambda^n \frac{c^\pm_{n-1}(x)}{n} + \lambda^{n-1} \frac{c^\pm_{n-2}(x)}{n-1} + \begin{cases} \mathrm{O}(\ln \lambda) & \text{if} \quad n = 2 \\ \mathrm{O}(\lambda^{n-2}) & \text{if} \quad n \geq 3. \end{cases}$$

$$[N'_\pm * \rho](x, \lambda) = \lambda^{n-1} c^\pm_{n-1}(x) + \lambda^{n-2} c^\pm_{n-2}(x) + \mathrm{O}(\lambda^{n-3}).$$

The setting of the problem

Weyl coefficients

- Classical:

$$c^{\pm}_{n-1}(x) = \frac{n}{(2\pi)^n} \sum_{j=\pm1}^{\pm m^{\pm}} \int_{\pm h^j(x,\xi)<1} d\xi.$$

- Chervova,Downes,Vassiliev'13:

$$c^{\pm}_{n-2}(x) = -\frac{n(n-1)}{(2\pi)^n} \sum_{j=\pm1}^{\pm m^{\pm}} \int_{\pm h^j(x,\xi)<1} \left[[v^j]^* a_0 v^j - \frac{\imath}{2}\left\{[v^j]^*, a_1 - h^j, v^j\right\} \right.$$

$$\left. + \frac{\imath}{n-1} h^j \left\{[v^j]^*, v^j\right\} \right] d\xi.$$

The setting of the problem

History

- Ivrii'80 - formula without proof
- Ivrii'82 - another formula with a 'proof'
- Rozemblyum'83 - similar formula
- Ivrii'84 - yet another formula without proof + algorithm
- Safarov'89 - formula with curvature term missing
- Chervova,Downes,Vassiliev'13 - answer by Levitan's method, propagator

Approximating powers of the resolvent

Approximating powers of the resolvent

Resolvent and powers

- $(A - z)^{-1} = \mathrm{Op}\left((a - z)^{\#,-1}\right) \mod \mathrm{L}^{-3}, \quad z \in \mathbb{C} \setminus \mathbb{R}$

Proposition (A.,Sjöstrand,Vassiliev'18)

For every $\nu \in \mathbb{N}$,

$$(A - z)^{-\nu} = \mathrm{Op}\left(\frac{1}{(n-1)!}\partial_z^{n-1}\left[(a - z)^{\#,-1}\right]\right) \mod \mathrm{L}^{-\nu-2}.$$

For every chart $\Omega \subset M$,

$$(A - z)^{-\nu}|_\Omega = \mathrm{Op}\left(\frac{1}{(n-1)!}\partial_z^{n-1}\left[(a - z)^{\#,-1}_{-1} + (a - z)^{\#,-1}_{-2}\right]\right)$$

$$+\mathrm{O}(|z|^{-\nu-2}) : H^0 \to H^{\nu+2}, \quad \forall z \in \mathbb{C} \setminus (\mathbb{D}(0,1) \cup \Gamma_\epsilon).$$

Approximating powers of the resolvent

Principal and subprincipal symbols

Proposition (A.,Sjöstrand,Vassiliev'18)

$$(a-z)^{\#,-1} = (a_1-z)^{-1} - (a_1-z)^{-1} a_0 (a_1-z)^{-1}$$

$$+\frac{\imath}{2}\left\{(a_1-z)^{-1}, a_1, (a_1-z)^{-1}\right\} \quad \text{mod } S^{-3}.$$

Ivrii'84:

$$(a_1-z)^{-1}\left\{a_1, (a_1-z)^{-1}, a_1\right\}(a_1-z)^{-1}$$

Approximating powers of the resolvent

Matrix trace

Spectral theorem $a_1 = \sum_j h^j P^j$.

Proposition (A.,Sjöstrand,Vassiliev'18)

$$\operatorname{tr}[(a-z)^{\#,-1}] = \sum_j \frac{1}{h^j - z} - \sum_j \frac{\operatorname{tr}[a_0 P^j]}{(h^j - z)^2}$$

$$+\frac{\imath}{2} \sum_{j,k,l} \frac{h^j - z}{(h^k - z)(h^l - z)} \operatorname{tr}\left\{P^k, P^j, P^l\right\} \quad \bmod S^{-3}.$$

Note: quintuple sum in Ivrii's form.

Approximating powers of the resolvent

Identity

Orthogonality $\quad P^j P^k = \delta^{jk} P^j$.

Proposition (A.,Sjöstrand,Vassiliev'18)

$$\operatorname{tr}\left\{P^k, P^j, P^l\right\} = 2\delta^{kj}\delta^{jl}\operatorname{tr}\left\{P^j, P^j, P^j\right\} - \delta^{kj}\operatorname{tr}\left\{P^l, P^j, P^l\right\}$$

$$-\delta^{jl}\operatorname{tr}\left\{P^k, P^j, P^k\right\} + \delta^{kl}\operatorname{tr}\left\{P^k, P^j, P^k\right\}.$$

Approximating powers of the resolvent

Matrix trace (cont.)

- Finally,

$$\operatorname{tr}[(a-z)^{\#,-1}] = \sum_j \frac{1}{h^j - z} - \sum_j \frac{\operatorname{tr}[a_0 P^j]}{(h^j - z)^2}$$

$$+\frac{\imath}{2} \sum_j \frac{\operatorname{tr}\left\{P^j, a_1 - h^j, P^j\right\}}{(h^j - z)^2} + \imath \sum_j \frac{\operatorname{tr}\left\{P^j, P^j, P^j\right\}}{h^j - z} \quad \operatorname{mod} S^{-3}.$$

Derivation of the explicit formula

Derivation of the explicit formula

Trace class versus singularity

Dilemma:

- $(A-z)^{-\nu}$ trace class iff $\nu \geq n+1$, $\nu \in \mathbb{N}$
- $(A-z)^{-\nu}$ singular on $z \in \mathbb{R}$ iff $\nu \leq n-1$

Salvation: consider

$$g_{n-1}(A,z) = \imath \left[\frac{2}{(A-z)^{n-1}} - \frac{1}{(A-2z)^{n-1}} - \frac{2}{(A-\bar{z})^{n-1}} + \frac{1}{(A-2\bar{z})^{n-1}} \right]$$

Proposition (A.,Sjöstrand,Vassiliev'18)

$g_{n-1}(A,z)$ is singular on $z \in \mathbb{R}$ and trace class on $z \in \mathbb{C} \setminus \mathbb{R}$.

Derivation of the explicit formula

Matrix trace of integral kernel

- Denote $f(x,z) = \operatorname{tr} g_{n-1}(A,z)(x,x), \quad \forall x \in M$
- $f(.,z) \in C(\mathrm{Vol}^{1/2}M), \quad \forall z \in \mathbb{C} \setminus \mathbb{R}$

$$
\begin{aligned}
f(x,z) = \sum_k g_{n-1}(\lambda_k; z)\|v_k(x)\|^2 = \int_0^{+\infty} g_{n-1}(\mu,z) N'_+(x,\mu)\, d\mu \\
-(-1)^n \frac{2^n - 1}{2^{n-1}} \imath \left[\frac{1}{z^{n-1}} - \frac{1}{\bar{z}^{n-1}}\right] \sum_{\lambda_k=0} \|v_k(x)\|^2 \\
-(-1)^n \int_0^{+\infty} g_{n-1}(\mu,z) N'_-(x,\mu)\, d\mu,
\end{aligned}
$$

where

$$
N'_\pm(x,\lambda) = \sum_{\pm\lambda_k > 0} \delta(\lambda - |\lambda_k|)\|v_k(x)\|^2.
$$

Derivation of the explicit formula

Mollified version

Denote

$$f^\rho(x,z) = \int_0^{+\infty} g_{n-1}(\mu,z)[N'_+ * \rho](x,\mu)d\mu$$
$$-(-1)^n \int_0^{+\infty} g_{n-1}(\mu,z)[N'_- * \rho](x,\mu)d\mu,$$

where

$$[N'_\pm * \rho](x,\lambda) = \sum_{\pm\lambda_k>0} \rho(\lambda - |\lambda_k|)\|v_k(x)\|^2.$$

Set $z = \lambda e^{\imath\varphi}, \quad \lambda > 1, \quad 0 < \varphi < \pi$.

Lemma (A.,Sjöstrand,Vassiliev'18)

$$f^\rho(x,\lambda e^{\imath\varphi}) - f(x,\lambda e^{\imath\varphi}) \xrightarrow[\lambda\to+\infty]{} 0.$$

Derivation of the explicit formula

Mollified version (cont.)

Lemma (A.,Sjöstrand,Vassiliev'18)

$$f^\rho(x, \lambda e^{\imath\varphi}) = b_1(x,\varphi)\lambda + b_0(x,\varphi) + \mathrm{o}(1) \quad \text{as} \quad \lambda \to +\infty,$$

where

$$b_1(x,\varphi) = -4\ln 2 \cdot (n-1)\sin\varphi \cdot \left[c^+_{n-1}(x) + (-1)^n c^-_{n-1}(x)\right],$$

$$b_0(x,\varphi) = -2\left[(\pi-\varphi)c^+_{n-2}(x) + (-1)^n \varphi c^-_{n-2}(x)\right].$$

Corollary (A.,Sjöstrand,Vassiliev'18)

$$f(x, \lambda e^{\imath\varphi}) = b_1(x,\varphi)\lambda + b_0(x,\varphi) + \mathrm{o}(1) \quad \text{as} \quad \lambda \to +\infty.$$

Derivation of the explicit formula

The other way

- Step 1:

$$f(x,z) = \imath\Big[2\,\mathrm{tr}[(A-z)^{-n+1}](x,x) - \mathrm{tr}[(A-2z)^{-n+1}](x,x)$$

$$-2\,\mathrm{tr}[(A-\bar{z})^{n-1}](x,x) + \mathrm{tr}[(A-2\bar{z})^{n-1}](x,x)\Big]$$

- Step 2:

$$\mathrm{tr}[(A-z)^{-n+1}](x,x) = \mathrm{Op}\bigg(\frac{1}{(n-2)!}\partial_z^{n-2}\,\mathrm{tr}\Big[(a-z)^{\#,-1}_{-1}$$

$$+(a-z)^{\#,-1}_{-2}\Big]\bigg)(x,x) + \mathrm{O}(|z|^{-n-1}), \quad \forall z \in \mathbb{C} \setminus (\mathbb{D}(0,1) \cup \Gamma_\epsilon)\,.$$

Derivation of the explicit formula

The other way (cont.)

- Step 3:

$$\mathrm{Op}(b)(x,x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} b(x,\xi) d\xi$$

- Step 4:

$$c^+_{n-2}(x) = -\frac{1}{2\pi} \lim_{\varphi \to 0+} b_0(x,\varphi)$$

For $c^-_{n-2}(x)$ replace A with $-A$.

Derivation of the explicit formula

Result

- Eigenprojections:

$$c^{\pm}_{n-2}(x) = -\frac{n(n-1)}{(2\pi)^n} \sum_{j=\pm 1}^{\pm m^{\pm}} \int_{\pm h^j(x,\xi)<1} \Bigg[\operatorname{tr}[a_0 P^j] + \frac{\imath}{2} \operatorname{tr}\left\{P^j, a_1 - h^j, P^j\right\} - \frac{\imath}{n-1} h^j \operatorname{tr}\left\{P^j, P^j, P^j\right\} \Bigg] d\xi.$$

- Eigenvectors:

$$c^{\pm}_{n-2}(x) = -\frac{n(n-1)}{(2\pi)^n} \sum_{j=\pm 1}^{\pm m^{\pm}} \int_{\pm h^j(x,\xi)<1} \Bigg[[v^j]^* a_0 v^j - \frac{\imath}{2} \left\{[v^j]^*, a_1 - h^j, v^j\right\} + \frac{\imath}{n-1} h^j \left\{[v^j]^*, v^j\right\} \Bigg] d\xi.$$

Thank you.