

The resolvent of a first order elliptic system and spectral asymptotics

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First order elliptic systems

First order elliptic systems

Weyl quantization on a manifold

- M - closed (i.e., compact without boundary) connected C^∞ manifold, $\dim M = n \geq 2$
- $\text{Vol}^{1/2}M$ - half-density bundle over M
- $A \in L^d(\text{Vol}^{1/2}M)$ is a Ψ DO of order $d \in \mathbb{R}$, i.e.,
- $A : C^\infty(\text{Vol}^{1/2}M) \rightarrow C^\infty(\text{Vol}^{1/2}M)$ s.t. \forall chart $M \subset \Omega \simeq \Omega' \subset \mathbb{R}^n$,

$$A \Big|_{C_c^\infty(\text{Vol}^{1/2}\Omega)} = \text{Op}(a_\Omega) + K_\Omega, \quad K_\Omega(\cdot, \cdot) \in C^\infty(\text{Vol}^{1/2}M \boxtimes \text{Vol}^{1/2}M).$$

$$a_\Omega \in S^d(T^*\Omega).$$

First order elliptic systems

Weyl quantization on a manifold (cont.)

- $M \supset \Omega_i \simeq \Omega'_i \subset \mathbb{R}^n$, $i = 1, 2$, two charts, $\varphi : \Omega_1 \cap \Omega_2 \rightarrow \Omega_1 \cap \Omega_2$ transition map,

$$a_{\Omega_1} = \varphi^* a_{\Omega_2} \quad \text{mod } S^{d-2}.$$

- Isomorphism $S^d/S^{d-2} \ni a \mapsto \text{Op}(a) \in L^d/L^{d-2}$
- a_d - principal symbol, a_{d-1} - subprincipal symbol,

$$a = a_d + a_{d-1} \quad \text{mod } S^{d-2}.$$

First order elliptic systems

First order system on a closed manifold

- $\mathbf{H} = \bigoplus_1^m L^2(\text{Vol}^{1/2}M)$ - intrinsic L^2 -space of m -columns
- $A : \bigoplus_1^m C^\infty(\text{Vol}^{1/2}M) \rightarrow \bigoplus_1^m C^\infty(\text{Vol}^{1/2}M)$ symmetric w. r. t. \mathbf{H}
- $A \in L^1(\bigoplus_1^m \text{Vol}^{1/2}M)$, a_1, a_0 Hermitian $m \times m$ matrices

$$A = \text{Op}(a_1 + a_0) \quad \text{mod } L^{-1}\left(\bigoplus_1^m \text{Vol}^{1/2}M\right)$$

First order elliptic systems

Elliptic and multiplicity-free

- A elliptic, i.e., $\det a_1(p) \neq 0$, $\forall p \in T^*M \setminus \{0\}$
- a_1 has simple spectrum, $\#\sigma(a_1(p)) = m$, $\forall p \in T^*M \setminus \{0\}$
- $M \supset \Omega \simeq \Omega' \subset \mathbb{R}^n$ local chart, $T^*\Omega \ni p = (x, \xi) \in \Omega \times \mathbb{R}^n$

First order elliptic systems

Eigenvalues and eigenfunctions of a_1

- Eigenvalues $\sigma(a_1(p)) = \{h^j(p)\}$, $j = -m^-, \dots, -1, 1, \dots, m^+$
- $m^+ + m^- = m$, $j \cdot h^j(p) > 0$, $\forall j$, $\forall p \in T^*M \setminus \{0\}$
- Eigenvectors

$$a_1(p)v^j(p) = h^j(p)v^j(p), \quad v^j \in C^\infty(T^*M, \mathbb{C}^m)$$

- Eigenprojections

$$P^j(p) = v^j(p) \times [v^j(p)]^*, \quad P^j \in C^\infty(T^*M, \mathfrak{gl}(m))$$

First order elliptic systems

Hamiltonian dynamics

- $h^j(p) = h^j(x, \xi)$ - Hamiltonian
- Trajectory $(x^j(t; y^j, \eta^j), \xi^j(t; y^j, \eta^j))$

$$\frac{dx^j(t)}{dt} = \partial_{\xi} h^j(x^j, \xi^j), \quad \frac{d\xi^j(t)}{dt} = -\partial_x h^j(x^j, \xi^j)$$

$$(x^j(0; y^j, \eta^j), \xi(0; y^j, \eta^j)) = (y^j, \eta^j)$$

The resolvent

The resolvent

Resolvent and powers

- $(A - z)^{-1} = \text{Op}((a - z)^{\#, -1}) \text{ mod } L^{-\infty}, \quad z \in \mathbb{C} \setminus \mathbb{R}$
- $(a - z)^{\#, -1}_{-1}$ - the principal symbol
- $(a - z)^{\#, -1}_{-2}$ - the subprincipal symbol
- $(A - z)^{-\nu} = \text{Op}((a - z)^{\#, -\nu}) \text{ mod } L^{-\infty}, \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad \nu \in \mathbb{N}$

Can compute $(a - z)^{\#, -\nu}$?

The resolvent

Resolvent and powers

Proposition (A., Sjöstrand, Vassiliev'20)

For every $\nu \in \mathbb{N}$,

$$\begin{aligned}(A - z)^{-\nu} &= \text{Op} \left(\frac{1}{(\nu - 1)!} \partial_z^{\nu-1} \left[(a - z)^{\#, -1} \right] \right) \quad \text{mod } L^{-\nu-2} \\ &= \text{Op} \left(\frac{1}{(\nu - 1)!} \partial_z^{\nu-1} \left[(a - z)_{-1}^{\#, -1} + (a - z)_{-2}^{\#, -1} \right] \right) \quad \text{mod } L^{-\nu-2} \\ &\quad \forall z \in \mathbb{C} \setminus (\mathbb{D}(0, 1) \cup \Gamma_\epsilon).\end{aligned}$$

The resolvent

Principal and subprincipal symbols

Proposition (A., Sjöstrand, Vassiliev'20)

$$(a - z)^{\#, -1} = (a_1 - z)^{-1} - (a_1 - z)^{-1} a_0 (a_1 - z)^{-1} \\ + \frac{i}{2} \left\{ (a_1 - z)^{-1}, a_1, (a_1 - z)^{-1} \right\} \pmod{S^{-3}}.$$

Ivrii'84:

$$(a_1 - z)^{-1} \left\{ a_1, (a_1 - z)^{-1}, a_1 \right\} (a_1 - z)^{-1}$$

$$\{a, b, c\} = \partial_x a \cdot b \cdot \partial_\xi c - \partial_\xi a \cdot b \cdot \partial_x c$$

The resolvent

Matrix trace

$(A - z)^{-\nu}$ is trace class iff $\nu \geq n + 1$, $n = \dim M$

- $$\operatorname{tr} [(A - z)^{-\nu}] = \operatorname{Op} \left(\operatorname{tr} [(a - z)^{\#, -\nu}] \right) \quad \text{mod } L^{-\nu-2}$$

- $$\operatorname{tr} [(A - z)^{-\nu}] = \operatorname{Op} \left(\frac{1}{(\nu - 1)!} \partial_z^{\nu-1} \operatorname{tr} [(a - z)^{\#, -1}] \right) \quad \text{mod } L^{-\nu-2}$$

The resolvent

Matrix trace

Spectral theorem $a_1 = \sum_j h^j P^j$.

Proposition (A., Sjöstrand, Vassiliev'20)

$$\begin{aligned} \operatorname{tr}[(a - z)^{\#, -1}] &= \sum_j \frac{1}{h^j - z} - \sum_j \frac{\operatorname{tr}[a_0 P^j]}{(h^j - z)^2} \\ &+ \frac{i}{2} \sum_{j,k,l} \frac{h^j - z}{(h^k - z)(h^l - z)} \operatorname{tr} \{ P^k, P^j, P^l \} \quad \text{mod } S^{-3}. \end{aligned}$$

Note: quintuple sum in Ivrii's form.

The resolvent

Identity

Orthogonality $P^j P^k = \delta^{jk} P^j$.

Proposition (A., Sjöstrand, Vassiliev'20)

$$\begin{aligned} \operatorname{tr} \{ P^k, P^j, P^l \} &= 2\delta^{kj} \delta^{jl} \operatorname{tr} \{ P^j, P^j, P^j \} - \delta^{kj} \operatorname{tr} \{ P^l, P^j, P^l \} \\ &\quad - \delta^{jl} \operatorname{tr} \{ P^k, P^j, P^k \} + \delta^{kl} \operatorname{tr} \{ P^k, P^j, P^k \}. \end{aligned}$$

The resolvent

Matrix trace (cont.)

- Finally,

$$\begin{aligned} \operatorname{tr}[(a - z)^{\#, -1}] &= \sum_j \frac{1}{h^j - z} - \sum_j \frac{\operatorname{tr}[a_0 P^j]}{(h^j - z)^2} \\ &+ \frac{i}{2} \sum_j \frac{\operatorname{tr}\{P^j, a_1 - h^j, P^j\}}{(h^j - z)^2} + i \sum_j \frac{\operatorname{tr}\{P^j, P^j, P^j\}}{h^j - z} \quad \text{mod } S^{-3}. \end{aligned}$$

Spectral asymptotics

Spectral asymptotics

Spectrum of A

- A self-adjoint with domain $\mathbf{D} = \bigoplus_1^m H^1(\text{Vol}^{1/2}M)$
- $\sigma(A)$ discrete
- $\sup \sigma(A) < +\infty$ iff $a_1 \leq 0$
- $\inf \sigma(A) > -\infty$ iff $a_1 \geq 0$

Spectral asymptotics

Local counting functions

- Spectrum $\sigma(A) = \{\lambda_k\}$, $k \in \mathbb{N}$
- Eigenfunctions $Av_k = \lambda_k v_k$, $v_k \in \mathbf{D}$
- Local counting functions $N_{\pm} \in C(\text{Vol}M) \otimes \mathcal{D}(\mathbb{R})'$

$$N_{\pm}(x, \lambda) = \begin{cases} 0 & \text{if } \lambda \leq 0 \\ \sum_{0 < \pm \lambda_k < \lambda} \|v_k(x)\|^2 & \end{cases}$$

- $N_{\pm}(x, \lambda) \nearrow$ as $\lambda \nearrow$ (step function)

Spectral asymptotics

Spectral asymptotics

- $y \in M$ non-focal if

$$\left| \left\{ \eta \in \mathbf{T}_y^* M \mid \|\eta\| = 1, \exists T > 0 \text{ s. t. } x^j(T; y, \eta) = y \right\} \right| = 0, \quad \forall j$$

Theorem (classical)

If $x \in M$ is non-focal then uniformly in $x \in M$

$$N_{\pm}(\lambda, x) = \lambda^n \frac{c_{n-1}^{\pm}(x)}{n} + \lambda^{n-1} \frac{c_{n-2}^{\pm}(x)}{n-1} + o(\lambda^{n-1}).$$

$$c_{n-1}^{\pm}, c_{n-2}^{\pm} \in C(\text{Vol}M)$$

Spectral asymptotics

Mollified spectral asymptotics

- $\rho \in \mathcal{S}(\mathbb{R})$ s. t. $\hat{\rho} \in C_c^\infty(-\mathbf{T}, \mathbf{T})$, $\hat{\rho}(0) = 1$, $\rho'(0) = 0$,

$$\mathbf{T} = \inf \left\{ T > 0 \mid \exists (y, \eta) \in \mathbf{T}^*M, \exists j \text{ s. t. } x^j(T; y, \eta) = y \right\} > 0.$$

Theorem (classical)

Uniformly in $x \in M$

$$[N_\pm * \rho](x, \lambda) = \lambda^n \frac{c_{n-1}^\pm(x)}{n} + \lambda^{n-1} \frac{c_{n-2}^\pm(x)}{n-1} + \begin{cases} O(\ln \lambda) & \text{if } n = 2 \\ O(\lambda^{n-2}) & \text{if } n \geq 3. \end{cases}$$

$$[N'_\pm * \rho](x, \lambda) = \lambda^{n-1} c_{n-1}^\pm(x) + \lambda^{n-2} c_{n-2}^\pm(x) + O(\lambda^{n-3}).$$

Spectral asymptotics

Weyl coefficients

- First Weyl coefficient (classical):

$$c_{n-1}^{\pm}(x) = \frac{n}{(2\pi)^n} \sum_{j=\pm 1}^{\pm m^{\pm}} \int_{\pm h^j(x,\xi) < 1} d\xi.$$

Spectral asymptotics

Weyl coefficients

- Second Weyl coefficient (Chervova, Downes, Vassiliev'13):

$$c_{n-2}^{\pm}(x) = -\frac{n(n-1)}{(2\pi)^n} \sum_{j=\pm 1}^{\pm m^{\pm}} \int_{\pm h^j(x, \xi) < 1} \left[[v^j]^* a_0 v^j - \frac{\imath}{2} \{ [v^j]^*, a_1 - h^j, v^j \} \right. \\ \left. + \frac{\imath}{n-1} h^j \{ [v^j]^*, v^j \} \right] d\xi$$

$$= -\frac{n(n-1)}{(2\pi)^n} \sum_{j=\pm 1}^{\pm m^{\pm}} \int_{\pm h^j(x, \xi) < 1} \left[\operatorname{tr}[a_0 P^j] + \frac{\imath}{2} \operatorname{tr} \{ P^j, a_1 - h^j, P^j \} \right. \\ \left. - \frac{\imath}{n-1} h^j \operatorname{tr} \{ P^j, P^j, P^j \} \right] d\xi.$$

Spectral asymptotics

History

- Ivrii'80 - formula without proof
- Ivrii'82 - another formula with a 'proof'
- Rozemblyum'83 - similar formula
- Ivrii'84 - yet another formula without proof + algorithm
- Safarov'89 - formula with curvature term missing
- Chervova, Downes, Vassiliev'13 - answer by Levitan's method, propagator

Spectral asymptotics

Derivation of the formula

Idea: Singularity of $\text{Tr} [(A - z)^{-\nu}]$ on \mathbb{R} .

Dilemma:

- $(A - z)^{-\nu}$ trace class iff $\nu \geq n + 1, \nu \in \mathbb{N}$
- $(A - z)^{-\nu}$ singular on $z \in \mathbb{R}$ iff $\nu \leq n - 1$

Spectral asymptotics

Derivation of the formula

Salvation: consider

$$g_{n-1}(A, z) = i \left[\frac{2}{(A - z)^{n-1}} - \frac{1}{(A - 2z)^{n-1}} - \frac{2}{(A - \bar{z})^{n-1}} + \frac{1}{(A - 2\bar{z})^{n-1}} \right]$$

Proposition (A., Sjöstrand, Vassiliev'20)

$g_{n-1}(A, z)$ is singular on $z \in \mathbb{R}$ and trace class on $z \in \mathbb{C} \setminus \mathbb{R}$.

Thus we look at the singularity of $\text{Tr} [g_{n-1}(A, z)]$ on \mathbb{R} .

Spectral asymptotics

Derivation of the formula

Knowing that 2-term asymptotics exists:

Proposition (A., Sjöstrand, Vassiliev'20)

$$\operatorname{tr} [g_{n-1}(A, \lambda \cdot e^{2\varphi})] (x, x) = b_1(x, \varphi)\lambda + b_0(x, \varphi) + o(1) \quad \text{as } \lambda \rightarrow +\infty,$$

$$c_{n-2}^+(x) = -\frac{1}{2\pi} \lim_{\varphi \rightarrow 0^+} b_0(x, \varphi).$$

Conclusion

Main results

- Formula for $(a - z)^{\#, -1} \pmod{S^{-3}}$, similar to Ivrii'84.

- Approximation of $(a - z)^{\#, -\nu}$ as

$$\frac{1}{(\nu - 1)!} \partial_z^{\nu-1} \left[(a - z)_{-1}^{\#, -1} + (a - z)_{-2}^{\#, -1} \right] \pmod{S^{-\nu-2}}$$

- Magic identity for $\text{tr}\{P^k, P^j, P^l\}$.
- Compact formula for $\text{tr} \left[(a - z)^{\#, -1} \right] \pmod{S^{-3}}$.
- Explicit formula for $c_{n-2}^{\pm}(x)$.

Thank you.

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