The resolvent of a first order elliptic system and spectral asymptotics

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Contents

First order elliptic systems

Weyl quantization on a manifold

- M - closed (i.e., compact without boundary) connected C^∞ manifold, $\dim M = n \geq 2$
- $\mathrm{Vol}^{1/2} M$ - half-density bundle over M
- $A \in \mathrm{L}^d(\mathrm{Vol}^{1/2} M)$ is a ΨDO of order $d \in \mathbb{R}$, i.e.,
- $A : C^\infty(\mathrm{Vol}^{1/2} M) \to C^\infty(\mathrm{Vol}^{1/2} M)$ s.t. $\forall$ chart $M \subset \Omega \simeq \Omega' \subset \mathbb{R}^n$,

$$A\Big|_{C_c^\infty(\mathrm{Vol}^{1/2}\Omega)} = \mathrm{Op}(a_\Omega) + K_\Omega, \quad K_\Omega(.,.) \in C^\infty(\mathrm{Vol}^{1/2} M \boxtimes \mathrm{Vol}^{1/2} M).$$

$$a_\Omega \in \mathrm{S}^d(\mathrm{T}^*\Omega).$$

First order elliptic systems

Weyl quantization on a manifold (cont.)

- $M \supset \Omega_i \simeq \Omega'_i \subset \mathbb{R}^n$, $i = 1, 2$, two charts, $\varphi : \Omega_1 \cap \Omega_2 \to \Omega_1 \cap \Omega_2$ transition map,

$$a_{\Omega_1} = \varphi^* a_{\Omega_2} \mod \mathrm{S}^{d-2}.$$

- Isomorphism $\mathrm{S}^d/\mathrm{S}^{d-2} \ni a \mapsto \mathrm{Op}(a) \in \mathrm{L}^d/\mathrm{L}^{d-2}$

- a_d - principal symbol, a_{d-1} - subprincipal symbol,

$$a = a_d + a_{d-1} \mod \mathrm{S}^{d-2}.$$

First order elliptic systems

First order system on a closed manifold

- $\mathbf{H} = \bigoplus_1^m L^2(\mathrm{Vol}^{1/2} M)$ - intrinsic L^2-space of m-columns
- $A : \bigoplus_1^m C^\infty(\mathrm{Vol}^{1/2} M) \to \bigoplus_1^m C^\infty(\mathrm{Vol}^{1/2} M)$ symmetric w. r. t. $\mathbf{H}$
- $A \in \mathrm{L}^1(\bigoplus_1^m \mathrm{Vol}^{1/2} M)$, a_1, a_0 Hermitian $m \times m$ matrices

$$A = \mathrm{Op}(a_1 + a_0) \mod \mathrm{L}^{-1}(\bigoplus_1^m \mathrm{Vol}^{1/2} M)$$

First order elliptic systems

Elliptic and multiplicity-free

- A elliptic, i.e., $\det a_1(p) \neq 0, \quad \forall p \in \mathrm{T}^*M \setminus \{0\}$
- a_1 has simple spectrum, $\sharp\sigma(a_1(p)) = m, \quad \forall p \in \mathrm{T}^*M \setminus \{0\}$
- $M \supset \Omega \simeq \Omega' \subset \mathbb{R}^n$ local chart, $\mathrm{T}^*\Omega \ni p = (x, \xi) \in \Omega \times \mathbb{R}^n$

First order elliptic systems

Eigenvalues and eigenfunctions of a_1

- Eigenvalues $\sigma(a_1(p)) = \{h^j(p)\}, \quad j = -m^-, \dots, -1, 1, \dots, m^+$
- $m^+ + m^- = m, \quad j \cdot h^j(p) > 0, \quad \forall j, \quad \forall p \in \mathrm{T}^* M \setminus \{0\}$
- Eigenvectors

$$a_1(p) v^j(p) = h^j(p) v^j(p), \quad v^j \in C^\infty(\mathrm{T}^* M, \mathbb{C}^m)$$

- Eigenprojections

$$P^j(p) = v^j(p) \times [v^j(p)]^*, \quad P^j \in C^\infty(\mathrm{T}^* M, \mathrm{gl}(m))$$

First order elliptic systems

Hamiltonian dynamics

- $h^j(p) = h^j(x, \xi)$ - Hamiltonian
- Trajectory $\big(x^j(t; y^j, \eta^j)\, , \, \xi^j(t; y^j, \eta^j) \big)$

$$\frac{dx^j(t)}{dt} = \partial_\xi h^j(x^j, \xi^j), \qquad \frac{d\xi^j(t)}{dt} = -\partial_x h^j(x^j, \xi^j)$$

$$\big(x^j(0; y^j, \eta^j)\, , \, \xi(0; y^j, \eta^j) \big) = (y^j\, , \, \eta^j)$$

The resolvent

The resolvent

Resolvent and powers

- $(A - z)^{-1} = \mathrm{Op}\left((a - z)^{\#,-1}\right) \mod \mathrm{L}^{-\infty}, \quad z \in \mathbb{C} \setminus \mathbb{R}$
- $(a - z)^{\#,-1}_{-1}$ - the principal symbol
- $(a - z)^{\#,-1}_{-2}$ - the subprincipal symbol
- $(A - z)^{-\nu} = \mathrm{Op}\left((a - z)^{\#,-\nu}\right) \mod \mathrm{L}^{-\infty}, \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad \nu \in \mathbb{N}$

Can compute $(a - z)^{\#,-\nu}$?

The resolvent

Resolvent and powers

Proposition (A.,Sjöstrand,Vassiliev'20)

For every $\nu \in \mathbb{N}$,

$$(A - z)^{-\nu} = \mathrm{Op}\left(\frac{1}{(\nu-1)!}\partial_z^{\nu-1}\left[(a-z)^{\#,-1}\right]\right) \quad \mathrm{mod}\ \mathrm{L}^{-\nu-2}$$

$$= \mathrm{Op}\left(\frac{1}{(\nu-1)!}\partial_z^{\nu-1}\left[(a-z)^{\#,-1}_{-1} + (a-z)^{\#,-1}_{-2}\right]\right) \quad \mathrm{mod}\ \mathrm{L}^{-\nu-2}$$

$$\forall z \in \mathbb{C} \setminus (\mathbb{D}(0,1) \cup \Gamma_\epsilon)\,.$$

The resolvent

Principal and subprincipal symbols

Proposition (A.,Sjöstrand,Vassiliev'20)

$$(a-z)^{\#,-1} = (a_1-z)^{-1} - (a_1-z)^{-1}a_0(a_1-z)^{-1}$$

$$+\frac{\imath}{2}\left\{(a_1-z)^{-1}, a_1, (a_1-z)^{-1}\right\} \quad \operatorname{mod} S^{-3}.$$

Ivrii'84:

$$(a_1-z)^{-1}\left\{a_1, (a_1-z)^{-1}, a_1\right\}(a_1-z)^{-1}$$

$\{a,b,c\} = \partial_x a \cdot b \cdot \partial_\xi c - \partial_\xi a \cdot b \cdot \partial_x c$

The resolvent

Matrix trace

$(A-z)^{-\nu}$ is trace class iff $\nu \geq n+1$, $n = \dim M$

- $$\mathrm{tr}\left[(A-z)^{-\nu}\right] = \mathrm{Op}\left(\mathrm{tr}\left[(a-z)^{\#,-\nu}\right]\right) \quad \bmod \mathrm{L}^{-\nu-2}$$

- $$\mathrm{tr}\left[(A-z)^{-\nu}\right] = \mathrm{Op}\left(\frac{1}{(\nu-1)!}\partial_z^{\nu-1}\mathrm{tr}\left[(a-z)^{\#,-1}\right]\right) \quad \bmod \mathrm{L}^{-\nu-2}$$

The resolvent

Matrix trace

Spectral theorem $\quad a_1 = \sum_j h^j P^j.$

Proposition (A.,Sjöstrand,Vassiliev'20)

$$\operatorname{tr}[(a-z)^{\#,-1}] = \sum_j \frac{1}{h^j - z} - \sum_j \frac{\operatorname{tr}[a_0 P^j]}{(h^j - z)^2}$$

$$+\frac{\imath}{2} \sum_{j,k,l} \frac{h^j - z}{(h^k - z)(h^l - z)} \operatorname{tr}\left\{P^k, P^j, P^l\right\} \quad \operatorname{mod} S^{-3}.$$

Note: quintuple sum in Ivrii's form.

The resolvent

Identity

Orthogonality $P^j P^k = \delta^{jk} P^j$.

Proposition (A.,Sjöstrand,Vassiliev'20)

$$\operatorname{tr}\left\{P^k, P^j, P^l\right\} = 2\delta^{kj}\delta^{jl}\operatorname{tr}\left\{P^j, P^j, P^j\right\} - \delta^{kj}\operatorname{tr}\left\{P^l, P^j, P^l\right\}$$

$$-\delta^{jl}\operatorname{tr}\left\{P^k, P^j, P^k\right\} + \delta^{kl}\operatorname{tr}\left\{P^k, P^j, P^k\right\}.$$

The resolvent

Matrix trace (cont.)

- Finally,

$$\mathrm{tr}[(a-z)^{\#,-1}] = \sum_j \frac{1}{h^j - z} - \sum_j \frac{\mathrm{tr}[a_0 P^j]}{(h^j - z)^2}$$

$$+\frac{\imath}{2}\sum_j \frac{\mathrm{tr}\left\{P^j, a_1 - h^j, P^j\right\}}{(h^j - z)^2} + \imath \sum_j \frac{\mathrm{tr}\left\{P^j, P^j, P^j\right\}}{h^j - z} \mod \mathcal{S}^{-3}.$$

Spectral asymptotics

Spectral asymptotics

Spectrum of A

- A self-adjoint with domain $\mathbf{D} = \bigoplus_{1}^{m} H^1(\mathrm{Vol}^{1/2} M)$
- $\sigma(A)$ discrete
- $\sup \sigma(A) < +\infty \quad \text{iff} \quad a_1 \leq 0$
- $\inf \sigma(A) > -\infty \quad \text{iff} \quad a_1 \geq 0$

Spectral asymptotics

Local counting functions

- Spectrum $\sigma(A) = \{\lambda_k\}, \quad k \in \mathbb{N}$
- Eigenfunctions $Av_k = \lambda_k v_k, \quad v_k \in \mathbf{D}$
- Local counting functions $N_\pm \in C(\mathrm{Vol} M) \otimes \mathcal{D}(\mathbb{R})'$

$$N_\pm(x, \lambda) = \begin{cases} 0 \quad \text{if} \quad \lambda \leq 0 \\ \sum_{0 < \pm\lambda_k < \lambda} \|v_k(x)\|^2 \end{cases}$$

- $N_\pm(x, \lambda) \nearrow$ as $\lambda \nearrow$ (step function)

Spectral asymptotics

Spectral aymptotics

- $y \in M$ *non-focal* if

$$\left|\left\{\eta \in \mathrm{T}^*_y M \,\middle|\, \|\eta\| = 1, \quad \exists T > 0 \quad \text{s. t.} \quad x^j(T; y, \eta) = y\right\}\right| = 0, \quad \forall j$$

Theorem (classical)

If $x \in M$ is non-focal then uniformly in $x \in M$

$$N_\pm(\lambda, x) = \lambda^n \frac{c^\pm_{n-1}(x)}{n} + \lambda^{n-1} \frac{c^\pm_{n-2}(x)}{n-1} + \mathrm{o}(\lambda^{n-1}).$$

$c^\pm_{n-1}, c^\pm_{n-2} \in C(\mathrm{Vol} M)$

Spectral asymptotics

Mollified spectral asymptotics

- $\rho \in \mathcal{S}(\mathbb{R})$ s. t. $\hat{\rho} \in C_c^\infty(-\mathbf{T}, \mathbf{T})$, $\quad \hat{\rho}(0) = 1$, $\quad \rho'(0) = 0$,

$$\mathbf{T} = \inf \left\{ T > 0 \middle| \quad \exists (y, \eta) \in \mathrm{T}^* M,\ \exists j \quad \text{s. t.} \quad x^j(T; y, \eta) = y \right\} > 0.$$

Theorem (classical)

Uniformly in $x \in M$

$$[N_\pm * \rho](x, \lambda) = \lambda^n \frac{c_{n-1}^\pm(x)}{n} + \lambda^{n-1} \frac{c_{n-2}^\pm(x)}{n-1} + \begin{cases} \mathrm{O}(\ln \lambda) & \text{if} \quad n = 2 \\ \mathrm{O}(\lambda^{n-2}) & \text{if} \quad n \geq 3. \end{cases}$$

$$[N'_\pm * \rho](x, \lambda) = \lambda^{n-1} c_{n-1}^\pm(x) + \lambda^{n-2} c_{n-2}^\pm(x) + \mathrm{O}(\lambda^{n-3}).$$

Spectral asymptotics

Weyl coefficients

- First Weyl coefficient (classical):

$$c_{n-1}^{\pm}(x) = \frac{n}{(2\pi)^n} \sum_{j=\pm 1}^{\pm m^{\pm}} \int_{\pm h^j(x,\xi)<1} d\xi.$$

Spectral asymptotics

Weyl coefficients

- Second Weyl coefficient (Chervova,Downes,Vassiliev'13):

$$c^{\pm}_{n-2}(x) = -\frac{n(n-1)}{(2\pi)^n} \sum_{j=\pm 1}^{\pm m^{\pm}} \int_{\pm h^j(x,\xi)<1} \left[[v^j]^* a_0 v^j - \frac{\imath}{2} \left\{ [v^j]^*, a_1 - h^j, v^j \right\} + \frac{\imath}{n-1} h^j \left\{ [v^j]^*, v^j \right\} \right] d\xi$$

$$= -\frac{n(n-1)}{(2\pi)^n} \sum_{j=\pm 1}^{\pm m^{\pm}} \int_{\pm h^j(x,\xi)<1} \left[\operatorname{tr}[a_0 P^j] + \frac{\imath}{2} \operatorname{tr} \left\{ P^j, a_1 - h^j, P^j \right\} - \frac{\imath}{n-1} h^j \operatorname{tr} \left\{ P^j, P^j, P^j \right\} \right] d\xi.$$

Spectral asymptotics

History

- Ivrii'80 - formula without proof
- Ivrii'82 - another formula with a 'proof'
- Rozemblyum'83 - similar formula
- Ivrii'84 - yet another formula without proof + algorithm
- Safarov'89 - formula with curvature term missing
- Chervova,Downes,Vassiliev'13 - answer by Levitan's method, propagator

Spectral asymptotics

Derivation of the formula

Idea: Singularity of $\mathrm{Tr}\,[(A-z)^{-\nu}]$ on $\mathbb{R}$.

Dilemma:

- $(A-z)^{-\nu}$ trace class iff $\nu \geq n+1$, $\nu \in \mathbb{N}$
- $(A-z)^{-\nu}$ singular on $z \in \mathbb{R}$ iff $\nu \leq n-1$

Spectral asymptotics

Derivation of the formula

Salvation: consider

$$g_{n-1}(A,z) = \imath\left[\frac{2}{(A-z)^{n-1}} - \frac{1}{(A-2z)^{n-1}} - \frac{2}{(A-\bar{z})^{n-1}} + \frac{1}{(A-2\bar{z})^{n-1}}\right]$$

Proposition (A.,Sjöstrand,Vassiliev'20)

$g_{n-1}(A,z)$ is singular on $z \in \mathbb{R}$ and trace class on $z \in \mathbb{C} \setminus \mathbb{R}$.

Thus we look at the singularity of $\mathrm{Tr}\,[g_{n-1}(A,z)]$ on $\mathbb{R}$.

Spectral asymptotics

Derivation of the formula

Knowing that 2-term asymptotics exists:

Proposition (A.,Sjöstrand,Vassiliev'20)

$$\operatorname{tr}\left[g_{n-1}(A, \lambda \cdot e^{\imath\varphi})\right](x,x) = b_1(x,\varphi)\lambda + b_0(x,\varphi) + \mathrm{o}(1) \quad \text{as} \quad \lambda \to +\infty,$$

$$c^{+}_{n-2}(x) = -\frac{1}{2\pi} \lim_{\varphi \to 0+} b_0(x,\varphi).$$

Conclusion

Main results

- Formula for $(a-z)^{\#,-1} \mod S^{-3}$, similar to Ivrii'84.
- Approximation of $(a-z)^{\#,-\nu}$ as
$$\frac{1}{(\nu-1)!}\partial_z^{\nu-1}\left[(a-z)^{\#,-1}_{-1}+(a-z)^{\#,-1}_{-2}\right] \mod S^{-\nu-2}$$
- Magic identity for $\operatorname{tr}\{P^k, P^j, P^l\}$.
- Compact formula for $\operatorname{tr}\left[(a-z)^{\#,-1}\right] \mod S^{-3}$.
- Explicit formula for $c^{\pm}_{n-2}(x)$.

Thank you.

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