# Elementary particle states in homogeneous cosmology

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Wigner elementary particle states

# Wigner elementary particle states

**Unitary representations of groups**

$G$ - locally compact type I group

**H** - separable complex Hilbert space

$\rho : G \to \mathcal{U}(\mathbf{H})$ - strongly cont. unitary rep.

$\hat{G} = \{\pi : G \to \mathbf{H}_\pi | \quad \pi \text{ unitary irrep}\} /[\simeq]$

# Wigner elementary particle states

**Unique decomposition**

$\forall\rho$ unirep, $\exists\hat{\mu}$ measure on $\hat{G}$, function $m : \hat{G} \to \mathbb{N}_0$ and unitary

$$\mathcal{F} : \mathbf{H} \to \int_{\hat{G}}^{\oplus} d\hat{\mu}(\pi) \bigoplus_{n_\pi} \mathbf{H}_\pi$$

such that

$$\mathcal{F} \circ \rho \circ \mathcal{F}^{-1} = \int_{\hat{G}}^{\oplus} d\hat{\mu}(\pi) \bigoplus_{n_\pi} \pi.$$

# Wigner elementary particle states

**Spectral theorem**

$H : \mathbf{D} \to \mathbf{H}$ self-adjoint, $\mathbf{D} \subset \mathcal{H}$ dense

Stone's theorem: $U_t = e^{\imath tH}$, strongly cont. unirep $U : \mathbb{R} \to \mathcal{U}(\mathbf{H})$

$\hat{\mathbb{R}} = \{\pi(x) = e^{\imath\pi x}, \quad \pi \in \mathbb{R}\}, \quad \mathbf{H}_\pi = \mathbb{C}$

$\exists\hat{\mu}, \quad \exists n_\pi, \quad \exists\mathcal{F} : \mathbf{H} \to \int_{\mathbb{R}}^{\oplus} d\hat{\mu}(\pi) n_\pi \mathbb{C}$ such that

$$\mathcal{F} \circ U_t \circ \mathcal{F}^{-1} = \int_{\mathbb{R}}^{\oplus} d\hat{\mu}(\pi) e^{\imath\pi t}, \quad \mathcal{F} \circ H \circ \mathcal{F}^{-1} = \int_{\mathbb{R}}^{\oplus} d\hat{\mu}(\pi)\pi$$

# Wigner elementary particle states

## Wigner's elementary particle concept

In quantum mechanics, $\forall f \in \mathbf{H}$, $\mathbb{C}f$ - proper (vector) state

If $\hat{\mu}$ discrete (e.g., $G$ compact) then

$$\mathcal{F}\mathbf{H} = \int_{\hat{G}}^{\oplus} d\hat{\mu}(\pi) \bigoplus_{n_\pi} \mathbf{H}_\pi = \bigoplus_{\hat{G}} \hat{\mu}(\pi) \bigoplus_{n_\pi} \mathbf{H}_\pi,$$

so that $\mathcal{F}^{-1}\mathbf{H}_\pi \subset \mathbf{H}$ subspace

Then $\forall f \in \mathbf{H}_\pi$, $\mathbb{C}f$ - proper elementary particle state with 'momentum' $\pi$

More generally, $\forall f \in \mathbf{H}_\pi$, $\mathbb{C}f$ - improper elementary particle state with 'momentum' $\pi$

# Wigner elementary particle states

## Eigenfunction expansion

$\mathbf{D} \subset \mathbf{H} \subset \mathbf{D}'$ Gelfand triple for $H$ self-adjoint

$\forall \pi \in \mathbb{R}$, $\exists(\Omega_\pi, \hat{\nu}_\pi)$ and $\{\xi_\omega\}_{\omega \in \Omega_\pi} \in \mathbf{D}'$ such that

$$\bigoplus_{n_\pi} \mathbf{H}_\pi \simeq \int_{\Omega_\pi}^{\oplus} d\hat{\nu}_\pi(\omega) \mathbb{C} \xi_\omega$$

$H\xi_\omega = \pi \xi_\omega$, $\omega \in \Omega_\pi$. Improper eigenfunctions.

For $H = \Delta$ on $\mathbf{H} = L^2(\mathbb{R})$, $\mathbf{D} = C_0^\infty(\mathbb{R})$,

$\xi_\omega(x) = e^{\imath \omega x} \in C^\infty \subset C_0^\infty(\mathbb{R})'$, $\omega \in \Omega_\pi = \{\pm \pi\}$

# Wigner elementary particle states

**Paley-Wiener theorem**

$M$ a $G$-homogeneous space, $\mu$ a left $G$-invariant measure

Gelfand triple $C_0^\infty(M) \subset L^2(M,\mu) \subset C_0^\infty(M)'$

How to decompose

$$C_0^\infty(M) \Leftrightarrow \left\{ \mathbf{D}_\pi \middle| \quad \pi \in \hat{G} \right\} ?$$

# Wigner elementary particle states

## Zonal functions

$M$ a $G$-homogeneous space, $\mu$ a left $G$-invariant measure

$H$ self-adjoint $G$-invariant operator with Gelfand triple $C_0^\infty(M) \subset L^2(M, \mu) \subset C_0^\infty(M)'$

Invariance of $H$ implies (formally) for $\forall \pi \in \hat{G}$,

$$\bigoplus_{n_\pi} \mathbf{H}_\pi \simeq \int_{\Omega_\pi}^{\oplus} d\hat{\nu}_\pi(\omega)\mathbb{C}\xi_\omega$$

For $M = G$ compact, $\hat{G}$ is discrete. Peter-Weyl theorem

$$\bigoplus_{\pi \in \hat{G}} \hat{\mu}(\pi) \bigoplus_{\omega \in \Omega_\pi} \hat{\nu}_\pi(\omega)\mathbb{C}\xi_\omega \subset C_0^\infty(G) = C^\infty(G)$$

is dense.

QM in homogeneous cosmology

# QM in homogeneous cosmology

**Homogeneous cosmological spacetimes**

Assume that

$M$ - smooth connected 4-dim Lorentzian manifold

$G = \mathrm{Iso}_0(M)$ - connected type I Lie group

$\forall m \in M$, the orbit $Gm \subset M$ an immersed 3-dim Riemannian submanifold

# QM in homogeneous cosmology

**Homogeneous cosmological spacetimes**

It follows that

$M \simeq \mathbb{R} \times \Sigma, \quad \{t\} \times \Sigma = G \cdot \{t\} \times \Sigma$ an orbit $\forall t \in \mathbb{R}$

Lorentzian metric $g(t, x) = dt^2 - h_t(x)$

$(\Sigma, h_t)$ - Riemannian $G$-homogeneous space $\forall t \in \mathbb{R}$

$M$ - globally hyperbolic spacetime

# QM in homogeneous cosmology

**Time separation in KG field**

Klein-Gordon field

$$(\Box + m^2)\varphi(x,t) = 0$$

$\Box = D_t - \Delta_t + m^2$

$H_t = -\Delta_t + m^2$ - a $G$-invariant Hamiltonian $\forall t \in \mathbb{R}$

$\mu_t = \det h_t = A(t)\mu_0$

# QM in homogeneous cosmology

## Time separation in KG field

$[H_t, H_{t'}] = 0$ on $C_0^\infty(\Sigma)$

$C_0^\infty(\Sigma) \subset L^2(\Sigma, \mu_0) \subset C_0^\infty(\Sigma)$

Eigenfunctions $\{\xi_\omega(x)\}_{\omega\in\hat{\Sigma}} \subset C^\infty(\Sigma)$, $\quad H_t\xi_\omega = \lambda_\omega(t)\xi_\omega$

Mode decomposition†

$$\varphi(x,t) = \int_{\hat{\Sigma}} d\hat{\mu}(\omega)\hat{\varphi}(\omega)T_\omega(t)\xi_\omega(x)$$

†*Z.A., 'A unified mode decomposition method for physical fields in homogeneous cosmology', Rev. Math. Phys. 26(3), 2014*

# QM in homogeneous cosmology

**Time separation in vacuum EM field**

Vacuum Maxwell's equations

$$dF = 0, \quad \delta F = 0, \quad F \in \Omega^2(M)$$

If $\mathrm{H}^1(\Sigma) = 0$ then $F = dA$, $A \in \Omega^1(M)$ and

$$\delta dA = 0$$

Lorentz gauge $\delta A = 0$,

$$\Box A = -(\delta d + d\delta)A = 0$$

# QM in homogeneous cosmology

**Time separation in Dirac field**

Vacuum Dirac equations

$$\begin{pmatrix} -\imath\nabla\!\!\!/ + m & 0 \\ 0 & \imath\nabla\!\!\!/ + m \end{pmatrix} \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = 0, \quad (\varphi, \psi) \in C^\infty(DM \oplus D^*M)$$

It follows

$$\begin{pmatrix} \Box_L + m^2 & 0 \\ 0 & \Box_L + m^2 \end{pmatrix} \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = 0$$

$\Box_L = \Box + \frac{1}{4}R$ - Lichnerowicz formula

# QM in homogeneous cosmology

**Dynamical elementary particle states**

$H_t \xi_\omega = \lambda_\omega(t) \xi_\omega$

If $\xi_\omega \in L^2(\Sigma, \mu_0)$ then proper eigenstate

$\mathbb{C}\xi_\omega$ - dynamical elementary particle state

Consistent in time

# QM in homogeneous cosmology

## Symmetry elementary particle states

$U : G \to \mathcal{U}(L^2(\Sigma, \mu_t))$ unirep

$$U_g f(x) = f(g^{-1}x), \quad \forall g \in G, \quad \forall x \in \Sigma, \quad \forall f \in L^2(\Sigma, \mu_t)$$

Fourier transform

$$\mathcal{F} L^2(\Sigma, \mu_t) = \int_{\hat{G}}^{\oplus} d\hat{\mu}_t(\pi) \bigoplus_{n_\pi} \mathbf{H}_\pi$$

$\forall f \in \mathbf{H}_\pi$, $\mathbb{C} f$ - symmetry elementary particle state with 'momentum' $\pi$

Consistent in time: $\hat{\mu}_t = A(t)\hat{\mu}_0$, $n_\pi = \mathrm{const}$

# QM in homogeneous cosmology

## Consistency of dynamics and symmetry

If $G$ compact (e.g., $\mathrm{SU}(2)$ or $\mathrm{SO}(3)$) then

Peter-Weyl theorem:

$$\overline{\bigoplus_{\pi\in\hat{G}} \mathbb{C}\left\{\xi_\omega \middle| \quad \omega \in \Omega_\pi\right\}} = C^\infty(G)$$

If $G = \mathrm{E}_+(3) = \mathrm{SO}(3) \rtimes \mathbb{R}^3$ then

$$\xi_\omega(x) = e^{\imath\omega\cdot x}, \quad \mathbf{H}_\pi \simeq \int_{|\omega|=\pi}^{\oplus} d\omega \mathbb{C} e^{\imath\omega\cdot x}$$

Results known for symmetric spaces (Harish-Chandra, Helgason etc.) and Bianchi models†

†*Z.A., R. Verch, 'Explicit harmonic and spectral analysis in Bianchi I-VII type cosmologies', Class. Quant. Grav. 30(15), 2013*

QFT in homogeneous cosmology

# QFT in homogeneous cosmology

**Symplectic representations**

Cauchy data $\mathbf{V}$. For KG field $\mathbf{V} = C_0^\infty(\Sigma) \oplus C_0^\infty(\Sigma)$

Conserved symplectic form $\sigma$ on $\mathbf{V}$ (Bosonic)

Dynamics: $V_t : \mathbb{R} \to \mathrm{Sp}(\mathbf{V}, \sigma)$

Symmetry: $V_g : G \to \mathrm{Sp}(\mathbf{V}, \sigma)$

# QFT in homogeneous cosmology

**Covariant quantization**

CCR quantization

$$(\mathbf{V}, \sigma) \to \mathcal{A}$$

$\mathrm{Sp}(\mathbf{V}, \sigma) \to \mathrm{Aut}(\mathcal{A})$

Dynamics: $\alpha_t : \mathbb{R} \to \mathrm{Aut}(\mathcal{A})$

Symmetry: $\alpha_g : G \to \mathrm{Aut}(\mathcal{A})$

# QFT in homogeneous cosmology

**Invariant equilibrium states**

Equilibrium state $\omega \circ \alpha_t = \omega$, $\forall t \in \mathbb{R}$

Invariant state $\omega \circ \alpha_g = \omega$, $\forall g \in G$

In the GNS rep $(\pi_\omega, \mathbf{H}_\omega, \Omega_\omega)$

$$\alpha_t(A) = U_t^* A U_t, \quad \alpha_g(A) = U_g^* A U_g, \quad \forall A \in \mathcal{A},$$

$$U_t : \mathbb{R} \to \mathcal{U}(\mathbf{H}_\omega), \quad U_g : G \to \mathcal{U}(\mathbf{H}_\omega)$$

# QFT in homogeneous cosmology

**Elementary particle states**

$U_t = e^{\imath tH}$, $H$ self-adjoint on $\mathbf{H}_\omega$

$[\alpha_t, \alpha_g] = 0$ hence $[U_g, H] = 0$

Decomposition

$$\mathcal{F}\mathbf{H}_\omega = \int_{\hat{G}}^{\oplus} d\hat{\mu}(\pi) \bigoplus_{n_\pi} \mathbf{H}_\pi$$

No natural Gelfand triple for $H$

Thank you.