On particle interpretation in homogeneous cosmology

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Contents

1. Minkowski spacetime (no gravity)
2. Homogeneous cosmological spacetimes (gravity)

Minkowski spacetime

Minkowski spacetime

Geometry

manifold $M = \mathbb{R}^4$ or $\mathbb{R}^{1,3} \simeq \mathbb{R} \times \mathbb{R}^3$, point (t, x)

Euclidean metric $dx^2 = dx_1^2 + dx_2^2 + dx_3^2$

Minkowski metric $g(t, x) = dt^2 - dx^2$

Isometry group $G = \mathrm{Iso}(M)_0 = \mathcal{P}_0^{1,3} = \mathbb{R}^4 \rtimes \mathrm{SO}_0(1, 3)$

Minkowski spacetime

Scalar field

Klein-Gordon equation: $(\Box + m^2)\varphi(t,x) = 0, \quad \Box = \partial_t^2 + \Delta$

Fourier transform or eigenfunction expansion:

$$\varphi(t,x) = \int_{\lambda(E,p)+m^2=0} d\mu(E,p)\hat{\varphi}(E,p)e^{\imath(Et+p\cdot x)},$$

$$\Box e^{\imath(Et+p\cdot x)} = \lambda(E,p)e^{\imath(Et+p\cdot x)}, \quad \lambda(E,p) = -E^2 + p^2.$$

Minkowski spacetime

Scalar field (2)

Spectral theorem: $\sigma(\Box) = \mathbb{R}$,

$$L^2(M) \simeq \int_{\sigma(\Box)}^{\oplus} d\lambda \int_{\lambda=\lambda(E,p)}^{\oplus} d\mu(E,p)\mathbb{C}e^{\imath(Et+p\cdot x)}.$$

$\mathbb{C}e^{\imath(Et+p\cdot x)}$ - *dynamical* elementary particle state with energy E and momentum p.

Minkowski spacetime

Wigner's particle concept

Ray reps of G $\Leftrightarrow$ unitary reps of $\widetilde{G} = \mathbb{R}_4 \rtimes \widetilde{\mathrm{SO}_0(1,3)}$

$\widetilde{\mathrm{SO}_0(1,3)} \simeq \mathrm{SL}(2,\mathbb{C}) \simeq \mathrm{Spin}(1,3)$

Harmonic analysis:

$$L^2(M) \simeq \int_{\widehat{\widetilde{G}}}^{\oplus} d\hat{\nu}(\pi)\mathbf{H}_\pi$$

$\widehat{\widetilde{G}}$ - unitary dual, $(\pi, \mathbf{H}_\pi)$ - unitary irrep

$\mathbb{C}f \subset \mathbf{H}_\pi$ - *symmetry* elementary particle state

Minkowski spacetime

Wigner's particle concept (2)

Little groups: $\widetilde{\mathrm{SO}(3)} \simeq \mathrm{SU}(2) \simeq \mathrm{Spin}(3)$, $\widetilde{\mathrm{E}_0(2)} \simeq \mathrm{Bi}(VII_0)$, $\widetilde{\mathrm{SO}_0(1,2)} \simeq \widetilde{\mathrm{SL}(2,\mathbb{R})}$ and $\mathrm{SL}(2,\mathbb{C})$

$$\widehat{\widetilde{G}} = 2\mathbb{R}_+ \times \widehat{\mathrm{SU}(2)} \quad \sqcup \quad \widehat{\widetilde{\mathrm{E}_0(2)}} \quad \sqcup \quad \widehat{\mathrm{SL}(2,\mathbb{C})} \quad \sqcup \quad 2\mathbb{R}_- \times \widehat{\widetilde{\mathrm{SL}(2,\mathbb{R})}}$$

Spin Casimir operator: $\mathsf{S} = |J \cdot p|^2 - |EJ - p \times K|^2$

$$L^2(M) \simeq \int_{\sigma(\Box)}^{\oplus} d\lambda \int_{\sigma(\mathsf{S})}^{\oplus} ds \mathbf{H}_{\lambda,s}$$

Minkowski spacetime

Correspondence

$$\mathbf{H}_{\lambda,s} \simeq \int_{\lambda,s}^{\oplus} d\mu(E,p)\mathbb{C}e^{\imath(Et+p\cdot x)}$$

Eigenfunction $\xi_{E,p}(t,x) = e^{\imath(Et+p\cdot x)}$,

$$\xi_{(E,p)}\left(g\cdot(t,x)\right) = \xi_{(E',p')}(t,x), \quad \forall g\in\widetilde{G}.$$

Homogeneous cosmological spacetimes

Homogeneous cosmological spacetimes

Geometry

Globally hyperbolic spacetime $M = \mathbb{R} \times \Sigma$, point (t, x).

Lorentzian metric $g(t, x) = dt^2 - h_t(x)$

Isometry group $G = \mathrm{Iso}_0(M) = \mathrm{Iso}_0(\Sigma, h_t)$ for all $t \in \mathbb{R}$

(Σ, h_t) - a G-homogeneous space

Homogeneous cosmological spacetimes

Scalar field

Klein-Gordon equation: $(\Box + m^2)\varphi(t,x) = 0, \quad \Box = \partial_t^2 + \Delta(t)$

Time-dependent spatial Fourier transform or eigenfunction expansion:

$$\varphi(t,x) = \int_{\tilde{\Sigma}} d\mu(p)\hat{\varphi}(p)T_p(t)X_p(x),$$

$$\Delta(t)X_p(x) = \lambda(p;t)X_p(x), \quad \ddot{T}_p(t) + \lambda(p;t)T_p(t) = 0.$$

Homogeneous cosmological spacetimes

Scalar field (2)

Spectral theorem: $L^2(M) \simeq \int_{\mathbb{R}}^{\oplus} dt\, L^2(\Sigma, h_t)$,

$$L^2(\Sigma, h_t) \simeq \int_{\sigma(\Delta(t))} d\lambda \int_{\lambda=\lambda(p;t)} d\mu(p) \mathbb{C} X_p(x)$$

$\mathbb{C} X_p(x)$ - *dynamical* elementary particle state with momentum p and variable energy.

Homogeneous cosmological spacetimes

Isotropic spacetimes

FRW spacetimes (classical):

$$\widetilde{G} = \begin{cases} \widetilde{\mathrm{E}_0(3)} = \mathbb{R}^3 \rtimes \widetilde{\mathrm{SO}(3)} \simeq \mathbb{R}^3 \rtimes \mathrm{SU}(2) \\ \mathrm{SO}(4) \simeq \widetilde{\mathrm{SU}(2)} \rtimes \mathrm{SU}(2) \\ \widetilde{\mathrm{SO}_0(1,3)} = \widetilde{\mathrm{SO}_0(1,2)} \rtimes \widetilde{\mathrm{SO}(3)} \simeq \widetilde{\mathrm{SL}(2,\mathbb{R})} \rtimes \mathrm{SU}(2) \end{cases}$$

LRS spacetimes (open): $\widetilde{G} = \mathrm{Bi}(N) \rtimes \widetilde{\mathrm{SO}(2)} \simeq \mathrm{Bi}(N) \rtimes \mathbb{R}$

Homogeneous cosmological spacetimes

Purely homogeneous spacetimes

Bianchi spacetimes: $\widetilde{G} = \mathrm{Bi}(N)$,

$$\mathrm{Bi}(I) = \mathbb{R}^3, \quad \mathrm{Bi}(II) = \mathbb{H}^{2+1}, \quad \mathrm{Bi}(III) = \mathrm{ax} + \mathrm{b}^{2+1} \quad \text{– classical}$$

$$\mathrm{Bi}(IV) - \mathrm{Bi}(VII) \quad - \quad \textit{recent!}^{\dagger}$$

$$\mathrm{Bi}(VIII) = \widetilde{\mathrm{SL}(2,\mathbb{R})}, \quad \mathrm{Bi}(IX) = \mathrm{SU}(2) \quad \text{– partially open}$$

† Z. A., R. Verch, 'Explicit harmonic and spectral analysis in Bianchi I-VII type cosmologies', Class. Quant. Grav. 30 (15), 2013

Homogeneous cosmological spacetimes

Wigner's particle concept

Harmonic analysis: $L^2(M) \simeq \int_{\mathbb{R}}^{\oplus} dt\, L^2(\Sigma, h_t)$,

$$L^2(\Sigma, h_t) \simeq \int_{\widehat{\widehat{G}}}^{\oplus} d\hat{\nu}(\pi)\mathbf{H}_\pi$$

$\mathbb{C}f \subset \mathbf{H}_\pi$ - *symmetry* elementary particle state

Correspondence:

$$\mathbf{H}_\pi \simeq \int_{\Sigma_\pi} d\mu(p) X_p(x)$$

Homogeneous cosmological spacetimes

More generally

G - connected type I Lie group, $\quad \nu$ - left Haar measure

$C_0^\infty(G) \subset L^2(G,\nu) \subset C_0^\infty(G)'$ - Gelfand triple

$L^2(G,\nu) \simeq \int^\oplus \mathbf{H}_\pi$. $\quad C_0^\infty(G) \leftrightarrow \{\mathbf{D}_\pi\}$?

G compact $\Rightarrow \hat{G}$ discrete, $C_0^\infty(G) = C^\infty(G)$, $\sigma(\Delta(t))$ distrete,
$\exists X_p \in C^\infty(G)$,

$$\mathcal{E} \doteq \bigoplus_{\pi\in\hat{G}} \bigoplus_{p\in\Sigma_\pi} \mathbb{C}X_p(x)$$

Peter-Weyl theorem: $\mathcal{E}$ dense in $C(G)$ and $C^\infty(G)$.

Thank you.