On the correspondence between harmonic analysis and spectral theory

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Outline

Physics: elementary particle states

Old and new examples

General approach

Physics: elementary particle states

Physics: elementary particle states

Quantum system

$\mathbf{H}$ - complex separable Hilbert space

$\mathcal{O} = \{T : \mathbf{H} \mapsto \mathbf{H}\}$ - observables, i.e., densely defined normal operators

$0 \neq \mathbb{C}f \subset \mathbf{H}$ - a state, i.e., a 1-dimensional Hilbert subspace

Physics: elementary particle states

Dynamics and symmetries

Dynamics: Distinguished observable $\mathsf{H} \in \mathcal{O}$, time variable $t \in \mathbb{R}$,

$$\frac{d\,\mathsf{T}}{dt} = \imath[\mathsf{H}, \mathsf{T}], \quad \forall\, \mathsf{T} \in \mathcal{O}.$$

Symmetries: A group G of unitary operators $\mathsf{U} \in \mathcal{O}$ such that

$$[\mathsf{U}, \mathsf{H}] = 0.$$

Physics: elementary particle states

Quantum mechanics

G - a real Lie group

M - a G-manifold

ν - a G-invariant measure (volume form)

$\mathbf{H} = L^2(M, \nu)$

$\mathrm{U}_g f(x) = f(g^{-1}x), \forall f \in \mathbf{H}, \forall g \in G$

H - a G-invariant Laplacian or Schrödinger operator

Physics: elementary particle states

What is an elementary particle state?

Given:

$\mathcal{S} = \{\mathsf{T} : \mathbf{H} \to \mathbf{H}\}$ - a semigroup of operators

Find:

$(\Omega, \hat{\nu})$ - a measure space

$\mathfrak{F} : \mathbf{H} \to \int_{\Omega}^{\oplus} d\hat{\nu}(\omega) \mathbf{H}_{\omega}$ - a unitary operator (Fourier transform)

$\mathbf{H}_{\omega}$ - irreducible invariant subspace under $\mathcal{S}$ for a.e. $\omega \in \Omega$

$\mathbb{C}f \subset \mathbf{H}_{\omega}$ - elementary particle states w.r.t. $\mathcal{S}$

Physics: elementary particle states

Dynamical particles = spectral theory

$$\mathcal{S} = \{\mathsf{H}^n, \quad n \in \mathbb{N}\}$$

$$\Omega \ni \omega = (\lambda, \omega_\lambda), \quad \lambda \in \sigma(\mathsf{H}), \quad \omega_\lambda \in \Omega_\lambda$$

$$\mathfrak{F} : \mathbf{H} \to \int_{\sigma(\mathsf{H})}^{\oplus} d\mu(\lambda) \int_{\Omega_\lambda}^{\oplus} \mathbf{H}_\omega$$

μ - spectral measure, $\mathsf{H}|_{\mathbf{H}_\omega} = \lambda \mathbf{1}$

$\mathbf{H}_\omega = \mathbb{C} f_\omega$ - wave function, spectral mode

Physics: elementary particle states

Symmetry particles = harmonic analysis/representation theory

$$\mathcal{S} = \{\mathsf{U}_g, \quad g \in G\}$$

$$\Omega \ni \omega = (\pi, \omega_\pi), \quad \pi \in \hat{G} \text{ irrep on } \mathbf{H}^\omega = \mathbf{H}_\pi, \quad \omega_\pi = 1, ..., d_\pi$$

$$\mathfrak{F} : \mathbf{H} \to \int_{\hat{G}}^{\oplus} d\hat{\nu}(\pi) \bigoplus_{\omega_\pi=1}^{d_\pi} \mathbf{H}^\omega$$

$\hat{\nu}$ - 'Plancherel' measure

$\mathbf{H}^\omega$ - Wigner's elementary particle states

Physics: elementary particle states

Question: What is an electron, a dynamical or a symmetry particle?

Physics: elementary particle states

Question: What is an electron, a dynamical or a symmetry particle?

Answer: It is both.

Old and new examples

Old and new examples

Example 1: Laplacian on the line.

$$M = \mathbb{R}, \quad d\nu(x) = dx, \quad \mathrm{H} = \Delta = -\partial_x^2, \quad \mathbf{H} = L^2(\mathbb{R}).$$

$$\mathcal{O} \ni \mathrm{X}, \mathrm{P}, \quad \mathrm{X}\, f(x) = xf(x), \quad \mathrm{P} = -\imath\partial_x f(x).$$

Spectral: $\sigma(\mathrm{H}) = [0, +\infty), \quad \mathbf{H} \simeq \int_{[0,+\infty)}^{\oplus} d\lambda \bigoplus_{\omega_\lambda = \pm 1} \mathbb{C} e^{\imath\lambda\omega_\lambda x}.$

$$G = \mathbb{R}, \quad \mathrm{U}_g\, f(x) = f(x - g), \quad \hat{G} = \mathbb{R}, \quad \mathbf{H}_\pi = \mathbb{C}, \quad \pi(g) = e^{\imath\pi g}.$$

Harmonic: $\mathbf{H} \simeq \int_{\mathbb{R}}^{\oplus} d\pi \mathbb{C} e^{\imath\pi x}.$

Old and new examples

Example 2: Laplacian on the plane.

$$M = \mathbb{R}^2, \quad d\nu(x) = dx^1 dx^2, \quad \mathrm{H} = \Delta = -\partial^2_{x^1} - \partial^2_{x^2}, \quad \mathbf{H} = L^2(\mathbb{R}^2).$$

$$\mathcal{O} \ni \mathrm{X}_i, \mathrm{P}_i, \quad \mathrm{X}_i f(x) = x^i f(x), \quad \mathrm{P}_i = -\imath\partial_{x^i} f(x), \quad i = 1, 2.$$

$$\textbf{Spectral:} \quad \mathbf{H} \simeq \int_{[0,+\infty)}^{\oplus} d\lambda\lambda \int_{\mathbb{S}^1}^{\oplus} dS(\omega_\lambda)\mathbb{C}e^{\imath(\lambda\omega_\lambda, x)}.$$

$$G = E(2) = \mathbb{R}^2 \rtimes U(1), \quad g = (y, \varphi), \quad gx = e^{\imath\varphi}x + y, \quad \hat{G} = \mathbb{R}_+.$$

$$\mathbf{H}_\pi = L^2(\mathbb{S}^1), \quad \pi(g)f(\psi) = e^{\imath\pi y^1} f(\psi - \varphi).$$

$$\textbf{Harmonic:} \quad \mathbf{H} \simeq \int_{\mathbb{R}_+}^{\oplus} d\pi\pi\mathbf{H}_\pi, \quad \pi(g)e^{\imath(\lambda\omega_\lambda, x)} = e^{\imath(\lambda\omega'_\lambda, x)}.$$

Old and new examples

Example 3: Laplacian on the sphere.

$$M = \mathbb{S}^2, \quad d\nu(\varphi) = \sin\varphi^1 d\varphi^1 d\varphi^2.$$

$$\mathrm{H} = \Delta = -\frac{1}{\sin\varphi^1}\partial_{\varphi^1}\sin\varphi^1\partial_{\varphi^1} - \frac{1}{\sin^2\varphi^1}\partial^2_{\varphi^2}, \quad \mathbf{H} = L^2(\mathbb{S}^2).$$

Spectral: $\sigma(\mathrm{H}) = \{l(l+1) | l \in \mathbb{N}_0\}, \quad \mathbf{H} \simeq \bigoplus_{l=0}^{\infty} \bigoplus_{m=-l}^{l} \mathbb{C} Y_l^m(\varphi).$

$$G = SO(3), \quad \hat{G} = \mathbb{N}_0,$$

Harmonic: $\mathbf{H} \simeq \bigoplus_{\pi=0}^{\infty} \mathbf{H}_\pi, \quad \mathbf{H}_\pi = \mathbb{C}\{Y_\pi^m(\varphi)\}_{m=-\pi}^{\pi}.$

Old and new examples

Example 4: Solvable Bianchi groups.

$$G = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ x^1 & e^{x^3 M} & \\ x^2 & & \end{pmatrix} \middle| \ (x^1, x^2, x^3) \in \mathbb{R}^3 \right\}, \quad g = (x^1, x^2, x^3),$$

$M(\mathrm{I})$	$M(\mathrm{II})$	$M(\mathrm{III})$	$M(\mathrm{IV})$	$M(\mathrm{V})$
0	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$M(\mathrm{VI}_q)$, $-1 < q \leq 1$, $q \neq 0$	$M(\mathrm{VII}_p)$, $p \geq 0$
$\begin{pmatrix} 1 & 0 \\ 0 & -q \end{pmatrix}$	$\begin{pmatrix} p & 1 \\ -1 & p \end{pmatrix}$

Old and new examples

Example 4: Solvable Bianchi groups (2).

Left and right generators

$$\begin{pmatrix} \mathrm{L}_1 \\ \mathrm{L}_2 \\ \mathrm{L}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ (x^1, x^2)M^\top & & 1 \end{pmatrix} \begin{pmatrix} \partial_{x^1} \\ \partial_{x^2} \\ \partial_{x^3} \end{pmatrix} \qquad \begin{pmatrix} \mathrm{R}_1 \\ \mathrm{R}_2 \\ \mathrm{R}_3 \end{pmatrix} = \begin{pmatrix} e^{x^3 M^\top} & & 0 \\ & & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \partial_{x^1} \\ \partial_{x^2} \\ \partial_{x^3} \end{pmatrix}$$

$$h^{-1} = h^{ij}\,\mathrm{R}_i \otimes \mathrm{R}_j$$

Left Haar measure

$$d\nu_h(x) = \sqrt{\det h_{**}}\, e^{-x^3 \operatorname{Tr} M} dx^1 dx^2 dx^3.$$

Old and new examples

Example 4: Laplacian on solvable Bianchi groups.

$$M = \mathbb{R}^3, \quad d\nu(x) = d\nu_h(x), \quad \mathbf{H} = L^2(\mathbb{R}^3, \nu_h).$$

$$\mathrm{H} = h^{ij}\,\mathrm{R}_i\,\mathrm{R}_j + \mathrm{Tr}\,M\,h^{3i}\,\mathrm{R}_i, \quad \sigma(\mathrm{H}) = [0, +\infty).$$

Spectral: $$\mathbf{H} \simeq \int_{[0,+\infty)}^{\oplus} d\lambda \int_{\mathbb{R}^2}^{\oplus} d\hat{\nu}(k^1) dk^2 e^{k^2\,\mathrm{Tr}\,M} \bigoplus_{\omega_\lambda=\pm 1} \mathbb{C}\xi_{k,h,\lambda,\omega_\lambda}(x).$$

$$\hat{G} \simeq \mathbf{R}^2 / e^{\mathbb{R}M^\top}, \quad U_g f(x) = f(g^{-1}x),$$

Harmonic: $$\mathbf{H} \simeq \int_{\hat{G}}^{\oplus} d\hat{\nu}(\pi) \bigoplus_{\omega_\pi=1}^{\infty} \mathbf{H}_\pi, \quad U_g \xi_{k,h,\lambda,\omega_\lambda}(x) = \xi_{k',h,\lambda,\omega_\lambda}(x)$$

$$k' = e^{g^3 M^\top} k, \quad g = (g^1, g^2, g^3) \in G.$$

General approach

General approach

Fourier transform

G - type I unimodular locally compact group, ν - Haar measure, $\hat{G}$ - unitary dual, $\hat{\nu}$ - Plancherel measure

$$\mathfrak{F} : L^2(G,\nu) \to \int_{\hat{G}}^{\oplus} d\hat{\nu}(\pi)\mathbf{H}_\pi \otimes \mathbf{H}_\pi^*,$$

$$\hat{f}(\pi) = \int_G d\nu(x) f(x)\pi(x), \quad f(x) = \int_{\hat{G}} d\hat{\nu}(\pi)\,\mathrm{Tr}[\pi^*(x)\hat{f}(\pi)],$$

Plancherel theorem:

$$\|f\|_2^2 = \int_{\hat{G}} d\hat{\nu}(\pi)\,\mathrm{Tr}[\hat{f}(\pi)^*\hat{f}(\pi)].$$

General approach

Fourier multipliers

$$\mathrm{U}_g f(x) = f(g^{-1}x), \quad f \in L^2(G,\nu), \quad g \in G.$$

$$\widehat{\mathrm{U}_g f}(\pi) = \pi(g)\hat{f}(\pi), \quad \widehat{f * h}(\pi) = \hat{f}(\pi)\hat{g}(\pi), \text{ where}$$

$$f * h(x) = \int_G d\nu(y) f(y) h(y^{-1}x).$$

$$\mathrm{H} \in \mathcal{B}(\mathbf{H}), \quad [\mathrm{U}_g, \mathrm{H}] = 0, \quad \forall g \in G, \text{ then}$$

$$\widehat{\mathrm{H} f} = \hat{f}(\pi)\,\mathrm{H}_\pi, \quad \mathrm{H}_\pi \in \mathcal{B}(\mathbf{H}_\pi), \text{ i.e.,}$$

$$\mathrm{H} \simeq \int_{\hat{G}}^{\oplus} d\hat{\nu}(\pi) \times \mathrm{H}_\pi\,.$$

General approach

Compact groups

$\hat{G}$ is discrete and $d_\pi = \dim \mathbf{H}_\pi < \infty$,

$$L^2(G, \nu) \simeq \bigoplus_{\hat{G}} \mathbf{H}_\pi \otimes \mathbf{H}_\pi^*.$$

If $\{e_j\}_{j=1}^{d_\pi}$ - eigenfunctions of H_π then

$\left\{\xi_{i,j}^\pi(x) = e_i^* \pi^*(x) e_j\right\}_{i,j=1}^{d_\pi}$ - eigenfunctions of H, $\quad C(G)$ functions.

Peter-Weyl theorem: $\mathbf{H}_\pi \otimes \mathbf{H}_\pi^* = \mathbb{C}\left\{\hat{\xi}_{i,j}^\pi\right\}$ and

$\mathbb{C}\left\{\hat{\xi}_{i,j}^\pi \middle| \quad \pi \in \hat{G}\right\}$ dense in $C(G)$.

General approach

Eigenfunction expansions

'... it is perhaps worth posing the general question of what conditions on Δ are needed for such a theory to exist (some hints in this direction are in Maurin'66)...' (R. Strichartz, 'Harmonic Analysis as Spectral Theory of Laplacians', 1989)

Gelfand triple: $\mathbf{D} \subset \mathbf{H} \subset \mathbf{D}'$, $\quad$ **D** - nuclear, $\mathrm{id} : \mathbf{D} \to \mathbf{H}$ continuous.

D - core of H, $\quad \mathrm{H} : \mathbf{D} \to \mathbf{D}$ continuous.

Eigenfunctions $\xi_{\lambda,\omega_\lambda} \in \mathbf{D}'$.

If H hypoelliptic, then $\xi_{\lambda,\omega_\lambda}$ regular.

General approach

Decomposition of continuity

$$\mathrm{T}_\pi \mapsto \mathrm{Tr}[\mathrm{T}_\pi\, \pi(x)], \quad L_1(\mathbf{H}_\pi) \simeq \mathcal{E}_\pi \subset C_b(G) \text{ (Godement'52).}$$

$$\mathcal{E} = \left\{ \int_{\hat{G}} d\hat{\nu}(\pi)\alpha(\pi), \quad \alpha(\pi) \in \mathcal{E}_\pi \right\} \subset C_b(G).$$

Generalized Peter-Weyl: $\mathcal{E}$ is dense in $C_b(G)$.

Generalized Bochner: $\widehat{C_b(G)}$ is the space of $\mathcal{E}_\pi$-valued finite measures on $\hat{G}$.

Thank you.